Rational Whitney tower filtration of links

Jae Choon Cha

Abstract. We present complete classifications of links in the 3-sphere modulo framed and twisted Whitney towers in a rational homology 4-ball. This provides a geometric characterization of the vanishing of the Milnor invariants of links in terms of Whitney towers. Our result also says that the higher order Arf invariants, which are conjectured to be nontrivial, measure the potential difference between the Whitney tower theory in rational homology 4-balls and that in the 4-ball extensively developed by Conant, Schneiderman and Teichner.

1. Introduction

Topology of dimension 4 is different from high dimensions because the Whitney move may fail. The essential problem is to find an embedded Whitney disk along which a pair of intersections of two sheets could be removed by a Whitney move. Once an immersed Whitney disk is obtained from fundamental group data, one may try to remove double points of the disk by finding a next stage of Whitney disks. Iterating this, we are led to the notion of a *Whitney tower*.

Since work of Cochran, Orr and Teichner [COT03], concordance of knots and links, which is the "local case" of general disk embedding, has been extensively studied via frameworks formulated in terms of Whitney towers. In this paper, we will focus on asymmetric Whitney towers in dimension 4 bounded by links in S^3, motivated from work of Conant, Schneiderman and Teichner [CST11, CST12c, CST14, CST12b, CST12a]. Whitney towers come in two flavors: *framed* and *twisted*. Whitney towers we consider have an *order*, which is a nonnegative integer measuring the number of iterated stages. Precise definitions can be found in Section 2.

The main result of this paper is a complete classification of links in S^3 modulo Whitney towers in *rational homology 4-balls*. To state our result, we use the following notation. Fix $m > 0$, and let $\overline{\mathbb{W}}^{\infty}_n$ be the set of m-component links in S^3 bounding a twisted Whitney tower of order n in a rational homology 4-ball with boundary S^3. We define the graded quotient $\overline{\mathsf{W}}^{\infty}_n$ of $\overline{\mathbb{W}}^{\infty}_n$ by the condition that L and L' in $\overline{\mathbb{W}}^{\infty}_n$ represent the same element in $\overline{\mathsf{W}}^{\infty}_n$ if and only if a band sum of L and $-L'$ lies in $\overline{\mathbb{W}}^{\infty}_{n+1}$. In fact, in Section 4.3, we will show that it is an equivalence relation, and $L \in \overline{\mathbb{W}}^{\infty}_{n+1}$ if and only if $[L] = 0$ in $\overline{\mathsf{W}}^{\infty}_n$. So we may write $\overline{\mathsf{W}}^{\infty}_n = \overline{\mathbb{W}}^{\infty}_n / \overline{\mathbb{W}}^{\infty}_{n+1}$.

Theorem A.
1. *Band sum is a well-defined operation on the set $\overline{\mathsf{W}}^{\infty}_n$, independent of the choice of bands, and $\overline{\mathsf{W}}^{\infty}_n$ is an abelian group under band sum.*
2. *$\overline{\mathsf{W}}^{\infty}_n$ is classified by the Milnor invariants of order n (= length $n+2$).*
3. *$\overline{\mathsf{W}}^{\infty}_n$ is a free abelian group of rank $m\mathcal{R}(m, n+1) - \mathcal{R}(m, n+2)$, where $\mathcal{R}(m,n) = \frac{1}{n}\sum_{d|n} \phi(d) \cdot m^{n/d}$ and $\phi(d)$ is the Möbius function.*

We remark that $\mathcal{R}(m,n)$ is the rank of the degree n part of the free Lie algebra on m variables, due to Witt (e.g., see [MKS66, Section 5.6]), and $m\mathcal{R}(m, n+1) - \mathcal{R}(m, n+2)$

2010 *Mathematics Subject Classification.* 57N13, 57N70, 57M25.

Key words and phrases. Whitney towers, Concordance, Rational homology 4-balls, Links, Milnor invariants, higher order Arf invariants.

is the number of linearly independent Milnor invariants of order n, due to Orr [Orr89]. The proof of Theorem A is given in Section 4.3. Especially see Theorem 4.6.

We also present a complete classification of links modulo *framed* Whitney towers. Briefly speaking, we define the framed analog $\overline{\mathbb{W}}_n$ and its graded quotient $\overline{\mathsf{W}}_n$ along the same lines using framed Whitney towers in rational homology 4-balls instead of twisted Whitney towers, so that $L \in \overline{\mathbb{W}}_{n+1}$ if and only if $[L] = 0$ in $\overline{\mathsf{W}}_n$. We prove that $\overline{\mathsf{W}}_n$ is an abelian group, and $\overline{\mathsf{W}}_n$ is completely classified by the Milnor invariants and the higher order Sato-Levine invariants introduced in [CST12c]. It turns out that $\overline{\mathsf{W}}_n$ is isomorphic to the direct sum of a certain determined number of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$. Details are given in Section 5. In particular see Theorem 5.1. We remark that even the proof that $\overline{\mathsf{W}}_n$ is an abelian group under band sum is not straightforward.

The above results remain true when we replace $\mathbb{Q}$ by any subring of $\mathbb{Q}$ in which 2 is invertible.

Theorem B. *For any subring R of $\mathbb{Q}$ containing $\frac{1}{2}$, a link in S^3 bounds a twisted Whitney tower of order n in an R-homology 4-ball if and only if the link bounds a twisted Whitney tower of order n in a rational homology 4-ball. The framed case analog holds too.*

We prove the twisted case of Theorem B in Section 4.2. In particular see Theorem 4.5. For the framed case, see Theorem 5.6 in Section 5.2.

Milnor invariants and rational Whitney towers. The problem of understanding the Milnor invariants geometrically has been addressed by numerous authors. Especially Igusa and Orr proved the *k-slice conjecture*, which asserts that a link L has vanishing Milnor invariants of length $\leq 2k$ if and only if L bounds disjoint surfaces in D^4 such that each loop on these surfaces can be pushed off to a loop lying in the kth lower central subgroup of the fundamental group of the complement of the surfaces [IO01]. A significantly strengthened version of the Igusa-Orr theorem was given in [CST14, Theorem 18] by Conant, Schneiderman and Teichner.

As a consequence of our main result, we present a geometric characterization of the vanishing of the Milnor invariants in terms of Whitney towers.

Theorem C. *A link L in S^3 has vanishing Milnor invariants of order $\leq n$ (or equivalently length $\leq n+2$) if and only if L bounds a twisted Whitney tower of order $n+1$ in a rational homology 4-ball.*

We remark that L bounds a twisted Whitney tower of order $n+1$ in a rational homology 4-ball if and only if L bounds a twisted capped grope of class $n+2$ in a rational homology 4-ball, due to [Sch06, Theorem 5] and [CST14, Lemma 23]. We prove Theorem C in Section 4.2 as a part of Theorem 4.5.

Higher order Arf invariants and rational Whitney towers. In their study of link concordance via Whitney towers in D^4, Conant, Schneiderman and Teichner introduced the *higher order Arf invariant* Arf_k ($k \geq 1$). Together with the Milnor invariants, Arf_k forms a complete set of invariants used to present classifications of links modulo twisted Whitney towers in D^4. Understanding the higher order Arf invariants, which remain mysterious yet, is the most significant open problem in the study of finite asymmetric Whitney towers. In particular the *higher order Arf invariant conjecture* asserts that Arf_k are nontrivial [CST12c, Conjecture 1.17].

Our main result provides a geometric interpretation of the (non-)vanishing of the higher order Arf invariants. Briefly, the higher order Arf invariants measure the difference between a bounding Whitney tower in the standard 4-ball and one in a rational homology 4-ball.

Theorem D. *For each $n \geq 0$, the following statements are equivalent.*

(1) $\mathrm{Arf}_k \equiv 0$ *for* $4k - 2 \leq n$.

(2) *A link $L \subset S^3$ bounds a twisted Whitney tower of order $n+1$ in D^4 if and only if L bounds a twisted Whitney tower of order $n+1$ in a rational homology 4-ball.*

We prove Theorem D at the end of Section 4.3. Especially see Corollary 4.8.

Some remarks on our approach. The proofs of our main results hinge, in an essential way, on the work of Conant, Schneiderman and Teichner on Whitney towers in D^4 [CST12c, CST14, CST12a, CST12b] which is summarized in [CST11].

They formulate algebraic analogs of the geometric theory of Whitney towers, in terms of intersection data of Whitney disks, and present complete classifications of the algebraic side using their proof of a conjecture of Levine [Lev01, Lev02]. To relate this to the geometric side, they prove a key result called the *order raising theorem* [CST12c, Theorems 1.9, 2.6, 2.10 and 4.4], whose origin goes back to [ST04, Theorem 2]. It essentially says that the vanishing of algebraic intersection data is sufficient to raise the order of a Whitney tower in D^4. This approach gives Whitney tower concordance classifications of links, modulo indeterminacy from a certain not-yet-understood part of the correspondence between the algebraic and geometric sides, which the higher order Arf invariant conjecture concerns.

A natural attempt for the study of Whitney towers in rational homology 4-balls, or more generally in general 4-manifolds, would be to develop a non-simply-connected version of the above algebraic theory and order rasing theorem. This appears to be a very interesting problem, whose solution seems far from being straightforward.

Instead, we present a different approach. We identify exactly which part of the Conant-Schneiderman-Teichner theory of Whitney towers in D^4 is annihilated in rational homology 4-balls. In fact we show that the information from the Milnor invariants (and the higher order Sato-Levine invariants in the framed odd order case) survives, while the higher order Arf invariants are eliminated when passed to the rational theory, as indicated in Theorem D. Put differently, the part not yet fully understood in the integral theory is exactly the information annihilated in the rational theory. This leads us to rational Whitney tower classification results *without indeterminacy*, as stated in Theorem A and Theorem 5.1.

To show that the Milnor invariant information is preserved in the rational theory, we first show a Milnor type theorem for Whitney towers in a rational homology 4-ball, which computes the lower central series quotients of the complement fundamental group. See Theorem 3.10. Using this and commutator calculus on a Whitney tower, we show that Milnor invariants (and higher order Sato-Levine invariants) are determined by a Whitney tower in a rational homology 4-ball. See Theorem 3.1. This generalizes an earlier result in [CST14].

The elimination of the higher order Arf invariants in the rational theory generalizes an earlier result too. Indeed, the figure eight knot, which has nontrivial Arf invariant, is known to bound a slice disk in a rational homology 4-ball [Cha07], and this tells us that the classical Arf invariant is not preserved under rational concordance. Our generalization to the higher order case is based on this. Precise formulations and proofs are given in Lemmas 4.3 and 5.4.

Organization of the paper. In Section 2, we review the definitions of Whitney towers and trees representing intersection data of Whitney disks. In Section 3, we investigate the relationship of the Milnor invariants of links and bounding Whitney towers in a rational homology 4-ball. In Section 4, we study twisted Whitney towers in a rational homology 4-ball. We give a complete characterization of links bounding a twisted Whitney tower

of a given order and prove Theorem A. Section 5 is devoted to the study of framed Whitney towers in a rational homology 4-ball.

Acknowledgement. The author thanks an anonymous referee for careful comments. This work was partially supported by NRF grant 2013067043.

2. Whitney towers and associated trees

In this section we will review definitions of twisted and framed asymmetric Whitney towers in 4-manifolds, and discuss uni-trivalent trees which arise naturally in the study of iterated intersections of surfaces, particularly for Whitney towers (e.g., see [Coc90, CT04a, CT04b, Sch06, CST07, CST12c, CST14]). Readers who are familiar with them may skip to Section 3, after reading this paragraph. In this paper a Whitney tower is always assumed to be union-of-disks-like (defined below), except the case of a Whitney tower concordance, which is union-of-annuli-like. Manifolds and immersed surfaces are always oriented.

2.1. Definitions of Whitney towers

In what follows, a *sheet* is an open subset of an immersed surface in a 4-manifold.

Definition 2.1 (Twisted and framed Whitney disk)**.** Suppose X is a 4-manifold and p, q are two intersections of opposite signs of two connected sheets A and B in X. A *Whitney circle* pairing p and q is an embedded circle α which is the union of an arc on A joining p and q and another arc on B joining p and q. A *Whitney disk* pairing p and q is an immersed disk D in X bounded by a Whitney circle α. We require that there is a collar neighborhood of ∂D in D whose intersection with $A \cup B$ is ∂D, while the complement of the collar is allowed to intersect the sheets.

For an immersed disk D, we call the restriction of the unique framing of D on ∂D the *disk framing*. On the boundary of a Whitney disk D, the tangential direction of one of the involved sheets and the common normal direction of D and the other sheet defines a framing, which we call the *Whitney framing*. Using $\mathrm{SO}(2) = \mathbb{Z}$, the disk framing with respect to the Whitney framing determines an integer $\omega(D)$ called the *twisting number* of D. If $\omega(D) = 0$, then D is called *framed*. When we do not require a disk to be framed in this sense, we call the disk *twisted*. (Technically a twisted Whitney disk may be framed.)

Definition 2.2 (Framed Whitney tower)**.** A *framed Whitney tower* in a 4-manifold X is a 2-complex defined inductively as follows. A union of properly immersed surfaces in X which are transverse to each other is a *framed Whitney tower*. Suppose T is a framed Whitney tower and D is an immersed framed Whitney disk in the interior of X pairing two intersections of opposite signs between two sheets in T. We allow the interior of D to transversely intersect the interior of surfaces and disks of T, but require D to be disjoint from the boundary of any surface or disk in T. Then T with D attached is a *framed Whitney tower*.

Definition 2.3 (Order)**.** The initial surfaces of a Whitney tower, namely those with boundary in ∂X, are called the *order 0 surfaces*. Inductively, an intersection of an order k sheet and an order ℓ sheet is called an *order $k+\ell$ intersection*. A Whitney disk pairing two order n intersections is called an *order $n+1$ disk*. A Whitney tower T is of *order n* if all intersections of order $< n$ are paired up by Whitney disks in T. (Intersections of order $\geq n$ are allowed to be unpaired.)

Definition 2.4 (Twisted Whitney tower)**.** A *twisted Whitney tower of order* n is defined exactly in the same way as a framed Whitney tower of order n, except that we allow Whitney disks of order $\geq \frac{n}{2}$ to be twisted. Disks of order $< \frac{n}{2}$ are still required to be framed.

A twisted Whitney tower of order n can be modified in such a way that all Whitney disks of order $> \frac{n}{2}$ are framed, by a boundary twist argument (see [CST12c, Section 4.1]). Using this, we always assume that a twisted Whitney tower of order $2k-1$ is indeed framed, and assume that a twisted Whitney disk of a twisted Whitney tower of order $2k$ has order k.

Following the convention of Freedman-Quinn [FQ90] used for gropes, we call a (framed or twisted) Whitney tower *union-of-disks-like* (respectively *union-of-annuli-like*) if each order zero surface is a disk (respectively an annulus). As mentioned at the beginning of this section, we assume that every Whitney tower is union-of-disks-like unless stated otherwise.

We remark that a Whitney tower can always be modified, using finger moves, in such a way that for each Whitney disk D (except the base disks or annuli) one of the following holds: (i) D is a twisted disk with $\omega(D) = \pm 1$, (ii) D is a framed disk with exactly one intersection point, or (iii) D is a framed disk with exactly two intersection points and they are paired by some other Whitney disk [CST12c, Lemma 2.12]. Such a tower is called *split.* We always assume that a Whitney tower is split, unless stated otherwise.

In this paper, links are always oriented and ordered.

Definition 2.5 (Boundary of Whitney towers)**.** Suppose X is a 4-manifold and L is a framed link in ∂X. We say that L *bounds an order* n *framed Whitney tower* T *in* X if (i) the boundary of the order zero disks of T is equal to L, and (ii) the unique framing of the order zero disks restricts to the given framing of L. For $n = 0$, we say that L *bounds an order* 0 *twisted Whitney tower* T *in* X if (i) holds, without requiring (ii). For $n > 0$, a framed link L *bounds an order* n *twisted Whitney tower* T *in* X if (i) and (ii) hold.

When a framed link L bounds an order 0 Whitney tower T, the *twisting number* $\omega(D) \in \mathbb{Z} = \mathrm{SO}(2)$ of an order 0 disk of T is defined to be the disk framing with respect to the given framing of L.

In Definition 2.5, we require the framing condition (ii) even for the twisted case when $n > 0$, because we always regard order $< \frac{n}{2}$ surfaces as framed, as we did in Definition 2.4. The same happens in the following definition.

Definition 2.6 (Whitney tower concordance)**.** Suppose X is a 4-manifold with $\partial X = \partial_+ X \sqcup -\partial_- X$. Two framed links $L \subset \partial_- X$ and $L' \subset \partial_+ X$ with m components are *order* n *framed Whitney tower concordant in* X if there is a union-of-annuli-like framed Whitney tower T of order n in X such that (i) T has m order zero annuli and the ith order zero annulus is cobounded by the ith component of L' and that of $-L$, and (ii) the framings of L and L' extend to the same framing of the order zero annuli. For $n > 0$, L and L' are *order* n *twisted Whitney tower concordant in* X if there is a twisted Whitney tower T of order n satisfying (i) and (ii). For $n = 0$, L and L' are *order* 0 *twisted Whitney tower concordant* if there is a twisted Whitney tower T satisfying (i).

Remark 2.7 (Framing of the boundary of rational Whitney towers)**.** In this paper we will mainly consider the case of a Whitney tower in a rational homology 4-ball bounded by S^3 (or standard D^4 as a special case) and a Whitney tower concordance in a rational homology $S^3 \times I$ bounded by $S^3 \times 1 \sqcup -S^3 \times 0$. Recall that the linking number of two knots in S^3 is equal to the algebraic intersection number of bounding immersed disks in a rational homology 4-ball bounded by S^3. The following basic observations are direct consequences of this fact and the above definitions.

(1) A framed link $L \subset S^3$ bounds an order 0 framed Whitney tower in a rational homology 4-ball if and only if each component of L is evenly framed, since an immersed disk in a rational homology 4-ball bounded by a knot $K \subset S^3$ induces an even framing on K. On the other hand, any framed link $L \subset S^3$ bounds an order 0 twisted Whitney tower in D^4.
(2) If a framed link $L \subset S^3$ bounds a framed/twisted Whitney tower of order $n \geq 1$ in a rational homology 4-ball, then the link is automatically zero framed and any two components have vanishing linking number; it follows from the fact that all the intersections of order zero disks are paired up by order 1 disks.
(3) If two framed links in S^3 are order $n \geq 1$ Whitney tower concordant in a rational homology $S^3 \times I$, then their framings are equal.

We will often say, e.g., "$L \subset S^3$ bounds an order $n \geq 1$ twisted/framed Whitney tower in a rational homology 4-ball" even when no framing on L is given. Using (2), this is understood as that L with the zero framing does.

2.2. Trees from intersection and twisting data

In this subsection we review a certain type of trees used in [CST12c, CST14]. Fix an integer $m \geq 1$. In this paper trees will always be uni-trivalent and oriented, that is, each vertex is either univalent or trivalent, and each trivalent vertex is endowed with a cyclic ordering of adjacent edges. As a convention, in a local planar diagram of a vertex and its adjancent edges, the edges are always ordered counterclockwise. A tree has *order* n if it has n trivalent vertices. A tree is *decorated* if each univalent vertex has a label in $\{1, \ldots, m\}$. For a rooted tree, namely when the tree has a distinguished univalent vertex, it is *decorated* if each non-root vertex has a label in $\{1, \ldots, m\}$. In this paper trees are always decorated. For two rooted trees t and t', the *inner product* $\langle t, t'' \rangle$ is defined by joining the roots of t and t'. The order of $\langle t, t'' \rangle$ is the sum of the orders of t and t''. Sometimes (but not always) we will label the root of a rooted tree by the symbol ∞; such a tree is called a *∞-tree.*

Suppose T is a twisted Whitney tower of order n. Fix an orientation of each disk in T, and fix an order of the order 0 disks. First, we associate to each disk D in T a rooted tree t_D as follows. For the ith order 0 disk D, define $t_D = \bullet\!\!-\!\!- i$, a rooted tree of order 0 with the non-root vertex labeled by i. For a twisted/framed Whitney disk D of order > 0, if D pairs two intersections between two disks D' and D'', define $t_D = \bullet\!\!-\!\!<^{t_{D''}}_{t_{D'}}$, that is, the rooted tree of order 1 with $t_{D'}$ and $t_{D''}$ attached to the leaves. Here, D' and D'' are chosen in such a way that if one travels along the Whitney circle ∂D near the involved negative intersection p of D' and D'', starting from D', passing through p and then entering into D'', then it agrees with the orientation of ∂D induced by the given orientation of D.

For each unpaired intersection p in the tower T, if $p \in D \cap D'$, then define $t_p = \langle t_D, t_{D'} \rangle$. (When we want to remember where the roots of t_D and $t_{D'}$ were, we draw the edge of t_p containing the original roots as $\succ\!\!-\!\circ\!\!-\!\!\prec$; the small enclosing circle denotes the location of p.) Note that t_D and D have the same order, and therefore so do t_p and p. For each intersection p, denote the sign of p by $\epsilon(p) = \pm 1$.

For each twisted Whitney disk D, let t^{∞}_D be the tree t_D with the root labeled by ∞, as a ∞-tree. Recall that $\omega(D)$ is the twisting number of D (see Definition 2.1).

Definition 2.8. For a twisted Whitney tower T of order n, define a formal sum $t^{\infty}_n(T)$ of trees by

$$t^{\infty}_n(T) = \sum_p \epsilon(p) \cdot t_p + \sum_D \omega(D) \cdot t^{\infty}_D$$

where p varies over the order n intersections and D varies over the twisted Whitney disks of order $n/2$. The second sum is regarded as vacuous if n is odd. Note that unpaired intersections of order $> n$ are ignored in $t_n^\infty(T)$.

3. Milnor invariants and rational Whitney towers

In this section we prove the following relationship of Milnor invariants of links and Whitney towers in rational homology 4-balls.

Theorem 3.1. *Suppose L is a framed link in S^3 bounding a twisted Whitney tower T of order $n \geq 0$ in a rational homolgy 4-ball bounded by S^3. Then the following hold.*

(1) *L has vanishing Milnor invariants of order $< n$ (or equivalently length $< n+2$).*
(2) *T determines the order n Milnor invariant of L. In fact, $\mu_n(L) = \eta_n(t_n^\infty(T))$.*

In Theorem 3.1 (2), μ_n denotes the total Milnor invariant of order n, and η_n denotes the summation map which was formulated in [Lev01, Lev02] and used extensively in [CST12c, CST14, CST12a]. We will review their definitions in Section 3.1.

Theorem 3.1 generalizes [CST12c, Theorem 6], which states the same conclusion under a weaker hypothesis that T is *in* D^4. We remark that the proof of [CST12c, Theorem 6] first converts the given Whitney tower to a capped grope and then works with the resulting grope, particularly using the grope duality of Krushkal and Teichner [KT97]. In our proof of Theorem 3.1, we present a Whitney tower argument inspired by the grope argument in [CST12c]. We wish this alternative approach, which works for Whitney towers in D^4 as well, to be a useful addition to the literature.

As a part of our proof of Theorem 3.1, we show a Milnor type theorem for Whitney towers in a rational homology 4-ball. See Theorem 3.10 in Section 3.3. Its analog for capped gropes in D^4 appeared earlier in [CST14, Lemma 33].

3.1. A quick review on the Milnor invariant and summation map

We begin by recalling the definition of the Milnor invariant and summation map, and setting up notations.

In the original work of Milnor [Mil57], the invariant is defined modulo certain indeterminancy to handle arbitrary links, but we will consider only the special case that it is well defined without indeterminancy.

Denote the lower central series of a group π by $\{\pi_k\}$, which is defined inductively by $\pi_1 = \pi$, $\pi_{k+1} = [\pi, \pi_k]$. In this paper, we use the convention $[a, b] = aba^{-1}b^{-1}$.

Suppose L is an m-component link in S^3 with $\pi = \pi_1(S^3 \smallsetminus L)$. Let $\mu_i \in \pi$ and $\lambda_i \in \pi$ be the class of a meridian and a zero linking longitude of the ith component respectively. Let F be a free group generated by $x_1, \ldots, x_m$. Let $F \to \pi$ be the meridian map defined by $x_i \mapsto \mu_i$. Suppose $n \geq 0$, and suppose λ_i is contained in π_{n+1}. (It is always the case for $n = 0$.) Then, by Milnor [Mil57, Theorem 4], $F \to \pi$ induces an isomorphism $\pi_{n+1}/\pi_{n+2} \cong F_{n+1}/F_{n+2}$. Let $w_i \in F_{n+1}/F_{n+2}$ be the image of λ_i under the isomorphism. The *Milnor invariant of length* $n+2$ can be defined to be the m-tuple $(w_1, \ldots, w_m) \in (F_{n+1}/F_{n+2})^m$. If the Milnor invariant of length $n+2$ vanishes, then the longitudes λ_i lie in π_{n+2}, so that the Milnor invariants of length $n+3$ can be defined.

Summarizing the above, the Milnor invariant of length $n+2$ is defined (without indeterminancy) when the Milnor invariants of length $\leq n+1$ vanish, and it is the case if and only if every longitude lies in the lower central subgroup π_{n+1}.

From the longitude elements $w_i \in F_{n+1}/F_{n+2}$, Milnor extracted numerical invariants denoted by $\overline{\mu}_L(i_1, \ldots, i_{n+1}, i)$ for $1 \leq i_j \leq m$, via the Magnus expansion. (This is why it is called of *length* $n+2$.) For our purpose, following [CST14, CST12c], it is convenient to

use the the free Lie algebra L generated by m variables $X_1, \dots, X_m$. We have $\mathsf{L} = \bigoplus_n \mathsf{L}_n$ where L_n is the degree n part; L_n is equal to the quotient of the free abelian group generated by n-fold brackets in $X_1, \dots, X_m$ modulo the Jacobi relation and alternativity relation $[X, X] = 0$. In particular, L_1 is the free abelian group generated by $X_1, \dots, X_m$. It is known that the association $x_i \mapsto X_i$ gives rise to an isomorphism $F_n/F_{n+1} \to \mathsf{L}_n$ which takes commutator brackets to Lie brackets. For instance see [MKS66, Section 5.7]. Let u_i be the image of w_i under $F_{n+1}/F_{n+2} \to \mathsf{L}_{n+1}$. The *total Milnor invariant of order* n is defined by

$$\mu_n(L) = \sum_{i=1}^{m} X_i \otimes u_i \in \mathsf{L}_1 \otimes \mathsf{L}_{n+1}.$$

Note that order n corresponds to length $n + 2$.

Let D_n be the kernel of the bracket map $\mathsf{L}_1 \otimes \mathsf{L}_{n+1} \to \mathsf{L}_{n+2}$ defined by $X \otimes Y \to [X, Y]$. Milnor's cyclic symmetry [Mil57, Theorem 5] implies that $\mu_n(L) \in \mathsf{D}_n$ for any link L. Moreover, as a function of the set of links L with $\mu_k(L) = 0$ for $k \leq n-1$, μ_n is surjective onto D_n. It is a consequence of [CST14, Theorem 6] and [Lev02, Theorem 1].

Remark 3.2 (Rank of D_n)**.** The range D_n of μ_n is a free abelian group of known rank. Due to Witt (e.g., see [MKS66, Section 5.6]), L_k is a free abelian group of rank $\mathcal{R}(m, k)$, where $\mathcal{R}(m, k) = \frac{1}{k} \sum_{d|k} \phi(d) \cdot m^{k/d}$ with $\phi(d)$ the Möbius function, as already given in Theorem A in the introduction. It follows that D_n is a free abelian group of rank

$$\mathcal{M}(m, n) := m\mathcal{R}(m, n+1) - \mathcal{R}(m, n+2).$$

It was first shown by Orr [Orr89] that $\mathcal{M}(m, n)$ is the number of linearly independent Milnor invariants of length $n + 2$ on links L with vanishing Milnor invariants of length $\leq n + 1$.

Remark 3.3 (Independence from meridian/longitude choices)**.** For any L with $\mu_q(L) = 0$ for $q \leq n - 1$, $\mu_n(L)$ is well-defined, independent of the choice of meridians μ_i (i.e., the meridian map $F \to \pi$). It is essentially because two meridians are conjugate: if a meridian map is given by $x_i \mapsto \mu_i$, then another meridian map is of the form $x_i \mapsto g_i \mu_i g_i^{-1}$, and it is straightforward to verify that they induce the same homomorphism $F_{n+1}/F_{n+2} \to \pi_{n+1}/\pi_{n+2}$, by using standard commutator calculus. Also, $\mu_n(L)$ is independent of the choice of longitudes λ_i, since any conjugate of $\lambda_i \in \pi_{n+1}$ is equal to λ_i itself modulo π_{n+2}.

Remark 3.4 (Milnor invariant for framed links)**.** For a *framed* link $L \subset S^3$, we define the Milnor invariant using pushoffs of components taken along the given framing, instead of zero linking longitudes. Then, for $n = 0$, $\mu_0(L)$ is equivalent to the pairwise linking numbers and framing of each component. (In the unframed case $\mu_0(L)$ is equivalent to the pairwise linking numbers.) In particular, L with $\mu_0(L) = 0$ is automatically zero framed. Since we always assume that $\mu_q(L) = 0$ for $q < n$ whenever we consider $\mu_n(L)$, it follows that there is no difference between framed and unframed cases for $\mu_n(L)$ with $n \geq 1$. Our definition for framed links will be useful in describing order 0 Whitney tower classifications of links in terms of μ_0.

We finish this subsection with the definition of the summation η_n which appeared in the statement of Theorem 3.1. Recall that a rooted tree t of order n decorated by $\{1, \dots, m\}$ determines a formal n-fold bracket in the variables $X_1, \dots, X_m$, which we denote by $B(t)$, in the standard manner: $B(\bullet\!\!-\, i\,) = X_i$, $B\big(\bullet\!\!-\!\!\!<^{t''}_{t'}\big) = [B(t'), B(t'')]$. We will often denote by $B(t)$ the element in L_{n+1} represented by the bracket $B(t)$. For a univalent vertex v of a tree t, let t_v be the rooted tree obtained by deleting the decoration of v and taking v as the root.

Definition 3.5 (Summation η_n)**.** For a tree t of order n, define $\eta_n(t) = \sum_v X_{\ell(v)} \otimes B(t_v) \in \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ where v varies over all the univalent vertices of t and $\ell(v)$ is the decoration of v. When n is even, define η_n for a ∞-tree t^∞ of order $\frac{n}{2}$ by $\eta_n(t^\infty) = \frac{1}{2}\eta_n(\langle t^\infty, t^\infty\rangle) \in \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$. It is straightforward to verify that $\eta_n(t^\infty)$ has integer coefficients. For a formal sum of decorated order n trees, and in addition order $\frac{n}{2}$ ∞-trees t^∞ when n is even, define η_n by extending the above linearly.

3.2. Computing meridians and Whitney circles in a Whitney tower

In this subsection we discuss how to compute Whitney circles and meridians of Whitney disks in the fundamental group of a Whitney tower complement using commutator calculus.

In what follows, the order 0 disks of a Whitney tower T in a 4-manifold X are always ordered. For a formal r-fold bracket B in $X_1, \dots, X_m$ with $r \geq k+1$, we also denote by the same symbol B the element in $\pi_1(X \smallsetminus T)_{k+1}/\pi_1(X \smallsetminus T)_{k+2}$ obtained by substituting a meridian of the ith order 0 disk for each occurance of X_i in the formal bracket B. This element is well-defined modulo $\pi_1(X \smallsetminus T)_{k+2}$, independent of the choice of a meridian, as in Remark 3.3. It is trivial in $\pi_1(X \smallsetminus T)_{k+1}/\pi_1(X \smallsetminus T)_{k+2}$ if $r > k+1$.

The following lemma says that the meridian of a Whitney disk D is essentially the commutator associated to the tree t_D.

Lemma 3.6 (Commutator expression of a meridian)**.** *Suppose T is a twisted Whitney tower in a 4-manifold X, and D is an order k disk in T. Then a meridian μ_D of D lies in $\pi_1(X \smallsetminus T)_{k+1}$ and $\mu_D = B(t_D)$ in $\pi_1(X \smallsetminus T)_{k+1}/\pi_1(X \smallsetminus T)_{k+2}$.*

Proof. We use an induction on k. For $k = 0$, the conclusion is straightforward. Suppose D is an order k disk with $k \geq 1$ and the conclusion holds for order $< k$. Since t_D has order k, $B(t_D)$ is a $(k+1)$-fold bracket. So it suffices to show that the meridian μ_D is of the form $B(t_D)$. The Whitney disk D pairs intersections of two disks D' and D'' of order r and s with $r + s + 1 = k$ by definition. The meridians $\mu_{D'}$ of D' and $\mu_{D''}$ of D'' are standard basis curves of a Clifford torus around the involved negative intersection. Since the Clifford torus meets D at a single transverse intersection, μ_D is equal to a commutator of $\mu_{D'}$ and $\mu_{D''}$. In fact, choosing D' and D'' in such a way that $t_D = \bullet\!\!-\!\!<\begin{smallmatrix} t_{D''} \\ t_{D'} \end{smallmatrix}$ holds (see the orientation convention in Section 2.2), we have $\mu_D = [\mu_{D'}, \mu_{D''}]$. Since $\mu_{D'} = B(t_{D'})$ and $\mu_{D''} = B(t_{D''})$ by the induction hypothesis, we have $\mu_D = [B(t_{D'}), B(t_{D''})] = B(t_D)$ as desired. $\square$

To compute the Whitney circles, we will use the following notations. Recall that we assume that a Whitney tower is split, and a twisted Whitney disk D in a Whitney tower of order n has order $n/2$ and $\omega(D) = \pm 1$.

Definition 3.7 (Complementary tree t^c_D of a Whitney disk D)**.** Suppose D is a Whitney disk in an order n twisted Whitney tower T. If D contains two paired intersections, proceed to the next stage Whitney disk that pairs the intersections. Repeating this, one eventually reaches either a framed Whitney disk with an unpaired intersection p of order $\geq n$, or a twisted Whitney disk D' of order $\frac{n}{2}$. Let $t^u_D := t_p$ in the former case and let $t^u_D := \langle t^\infty_{D'}, t^\infty_{D'}\rangle$ in the latter case. Our t^u_D contains t_D as a subtree; the root of t_D is the midpoint of an edge of t^u_D. When $t^u_D = \langle t^\infty_{D'}, t^\infty_{D'}\rangle$, we just fix one of the two copies of t_D in t^u_D. Define the *complementary tree* t^c_D of D to be t^u_D with t_D removed, with the root of t_D as the root of t^c_D. Define the *complementary sign* ϵ^c_D to be the sign $\epsilon(p)$ of p if $t^u_D = t_p$, and to be the twisting number $\omega(D')$ if $t^u_D = \langle t^\infty_{D'}, t^\infty_{D'}\rangle$.

If D is an order k disk in a Whitney tower of order n, then the complementary tree t^c_D of has order $\geq n - k$, since t^u_D has order $\geq n$.

Lemma 3.8 (Commutator expression of a Whitney circle)**.** *Suppose T is an order n twisted Whitney tower in a 4-manifold X and D is an order k Whitney disk in T with $k > 0$. Let γ_D be a pushoff of the Whitney circle ∂D, taken along the Whitney framing. Then γ_D lies in $\pi_1(X \smallsetminus T)_{n-k+1}$, and $\gamma_D = B(t^c_D)^{\epsilon^c_D}$ in $\pi_1(X \smallsetminus T)_{n-k+1}/\pi_1(X \smallsetminus T)_{n-k+2}$.*

It follows from Lemma 3.8 that γ_D is trivial in $\pi_1(X \smallsetminus T)_{n-k+1}/\pi_1(X \smallsetminus T)_{n-k+2}$ if the complementary tree t^c_D has order $> n-k$, or equivalently t^u_D has order $> n$.

Proof of Lemma 3.8. Let $G = \pi_1(X \smallsetminus T)$. As a special case, suppose D is a framed disk which has an order $\geq n$ unpaired intersection p with another disk D'. Then $\gamma_D = (\mu_{D'})^{\epsilon_p} = B(t_{D'})^{\epsilon^c_D}$ by Lemma 3.6. Since $t^u_D = t_p = \langle t_D, t_{D'}\rangle$, $t^c_D = t_{D'}$. It follows that $\gamma_D = B(t^c_D)^{\epsilon^c_D}$ as claimed. As another special case, suppose D is a twisted Whitney disk. Then since γ_D is taken along the Whitney framing, $\gamma_D \cdot (\mu_D)^{-\omega(D)}$ bounds a parallel of D. (Note that the exponent $-\omega(D)$ represents the disk framing with respect to the Whitney framing, since $\omega(D) \in \mathbb{Z} = \mathrm{SO}(2)$ is defined to be the Whitney framing with respect to the disk framing.) Therefore $\gamma_D = (\mu_D)^{\omega(D)} = B(t_D)^{\epsilon^c_D}$ by Lemma 3.6. Since $t^u_D = \langle t^\omega_D, t^\omega_D\rangle$, t^c_D is t_D itself. It follows that $\gamma_D = B(t^c_D)^{\epsilon^c_D}$.

Now we proceed inductively, from higher to lower stage Whitney disks, using the above cases as the initial step. Suppose D is a Whitney disk of order k. If D is not one of the above two special cases, then D is a framed disk with two intersections with another disk D' and the intersections are paired by a next stage Whitney disk D''. The induction hypothesis is that $\gamma_{D''} = B(t^c_{D''})^{\epsilon^c_{D''}}$.

We have either $t_{D''} = \bullet\!\!-\!\!\!<^{t_{D'}}_{t_D}$ or $\bullet\!\!-\!\!\!<^{t_D}_{t_{D'}}$. We will present details only for the former case, since the argument applies to the latter case in the essentially same way. Figure 1 shows the disks D, D' and D'' when $t_{D''} = \bullet\!\!-\!\!\!<^{t_{D'}}_{t_D}$. The circular arrows near ∂D and $\partial D''$ specify the orientations of D and D''. The disk D' is oriented in such a way that $\mu_{D'}$ is a positively oriented meridian. In Figure 1, the negatively oriented meridian of D' which is near the $-$ intersection is equal to $\gamma_{D''}\mu^{-1}_{D'}\gamma^{-1}_{D''}$. (Here one may use a basepoint near the $+$ intersection.) Therefore the pushoff γ_D of ∂D is the product of $\mu_{D'}$ and $\gamma_{D''}\mu^{-1}_{D'}\gamma^{-1}_{D''}$. By Lemma 3.6 and the induction hypothesis, $\gamma_D = [\mu_{D'}, \gamma_{D''}] = [B(t_{D'}), B(t^c_{D''})^{\epsilon^c_{D''}}]$. Using $\epsilon^c_{D''} = \epsilon^c_D$, and using $[a, b^{-1}] = b^{-1}[a,b]^{-1}b$ when $\epsilon^c_D = -1$, we obtain $\gamma_D = [B(t_{D'}), B(t^c_{D''})]^{\epsilon^c_D}$ in G_{n-k+1}/G_{n-k+2}. Since $t_{D''} = \bullet\!\!-\!\!\!<^{t_{D'}}_{t_D}$, the complementary tree of D is given by $t^c_D = \bullet\!\!-\!\!\!<^{t^c_{D''}}_{t_{D'}}$. It follows that $B(t^c_D)^{\epsilon^c_D} = [B(t_{D'}), B(t^c_{D''})]^{\epsilon^c_D} = \gamma_D$ as promised. $\square$

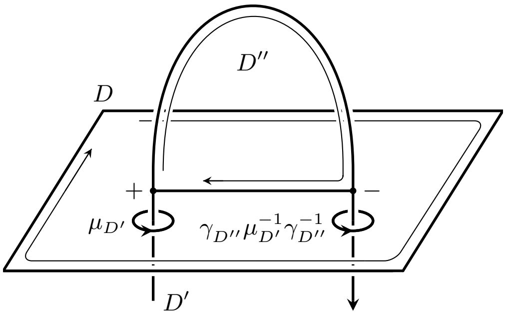

FIGURE 1. The disks D, D' and D'' and the meridian $\mu_{D'}$.

The proof of Lemma 3.8 applies to an order 0 disk D in essentially the same way. The statement is as follows.

Lemma 3.9. *Suppose T is a twisted Whitney tower of order $n \geq 0$ in a 4-manifold X bounded by a framed link L. Then the ith longuitude λ_i of L taken along the given framing lies in $\pi_1(X \smallsetminus T)_{n+1}$. Furthermore, if the formal sum $t^{\backsim}_n(T)$ is of the form*

$$t^{\backsim}_n(T) = \sum_t a(t) \cdot t + \sum_{t^{\backsim}} b(t^{\backsim}) \cdot t^{\backsim}$$

with $a(t)$, $b(t) \in \mathbb{Z}$, then

$$\lambda_i = \left(\prod_t \prod_{\substack{v \in t \\ \ell(v)=i}} B(t_v)^{a(t)} \right) \cdot \left(\prod_{t^{\backsim}} \prod_{\substack{u \in t^{\backsim} \\ \ell(u)=i}} B(\langle t^{\backsim}, t^{\backsim} \rangle_u)^{b(t^{\backsim})} \right) \text{ in } \frac{\pi_1(X \smallsetminus T)_{n+1}}{\pi_1(X \smallsetminus T)_{n+2}}.$$

Here t varies over order n trees appearing in $t^{\backsim}_n(T)$, v varies over the univalent vertices of t with decoration $\ell(v) = i$, $t^{\backsim}$ varies over order $\frac{n}{2}$ $\backsim$-trees appearing in $t^{\backsim}_n(T)$, and u varies over univalent vertices of a fixed copy of $t^{\backsim}_D$ in $\langle t^{\backsim}_D, t^{\backsim}_D \rangle$ with label $\ell(u) = i$.

Recall that for a tree t and its univalent vertex v, t_v is the rooted tree obtained by deleting the label of v and taking v as the root, as we did in Definition 3.5.

Proof. Let D be the ith order 0 disk of T. For each order n unpaired intersection on D, choose a disk neighborhood in D. For each pair of opposite intersections of D and another disk D' which are paired by a next stage disk D'', choose a disk neighborhood of the arc $D \cap D''$ in D. Denote these disk neighborhoods by $U_1, U_2, \ldots$; so each U_j contains either an order n unpaired intersection or an arc of the form $D \cap D''$. We may assume that the subdisks U_j are mutually disjoint. For each U_j, a pushoff γ_j of ∂U_j is computed by the argument of Lemma 3.8. Here, instead of the tree t^u_D used in Lemma 3.8, we use either an order n tree t appearing in $t^{\backsim}_n(T)$, or $\langle t^{\backsim}, t^{\backsim} \rangle$ for some order $\frac{n}{2}$ $\backsim$-tree $t^{\backsim}$ appearing in $t^{\backsim}_n(T)$, which has a univalent vertex v with label $\ell(v) = i$. Then t_v or $\langle t^{\backsim}, t^{\backsim} \rangle_v$ plays the role of the complementary tree. Therefore, by the argument of Lemma 3.8, we obtain $\gamma_j = B(t_v)^{\pm 1}$ or $B(\langle t^{\backsim}, t^{\backsim} \rangle_v)^{\pm 1}$, where $\pm$ is the sign of the coefficient of t or $t^{\backsim}$. When $n > 0$, each univalent vertex v of a tree appearing in $t^{\backsim}_n(T)$ with $\ell(v) = i$ is involved in the computation of exactly one γ_j. Since a pushoff of the boundary of D is equal to $\prod_j \gamma_j$, the promised formula for λ_i follows. When $n = 0$, the situation is indeed simpler but a minor change is needed. All intersections on D are unpaired and of order 0, and in addition, there may be trees of the form $t^{\backsim} = \backsim\!\!-\!\!-\, i$ in $t^{\backsim}(T)$, which is not involved in the computation for any D_j but yields a $\pm$ twisting for D. Nonetheless, the contribution of such a twisting to λ_i is $\mu_D^{\pm 1} = B(\bullet\!\!-\!\!-\, i)^{\pm 1} = B(\langle t^{\backsim}, t^{\backsim} \rangle_v)^{\pm 1}$, where μ_D is a meridian of the ith order zero disk D. Therefore the claimed formula for λ_i holds. $\square$

3.3. A Milnor type theorem for rational Whitney towers

Define the *rational lower central subgroups* $G^{\mathbb{Q}}_k$ ($k \geq 1$) of a group G by $G^{\mathbb{Q}}_1 := G$ and

$$G^{\mathbb{Q}}_{k+1} = \operatorname{Ker} \left\{ G^{\mathbb{Q}}_k \longrightarrow \frac{G^{\mathbb{Q}}_k}{[G, G^{\mathbb{Q}}_k]} \longrightarrow \frac{G^{\mathbb{Q}}_k}{[G, G^{\mathbb{Q}}_k]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \right\}.$$

It is straightforward to verify that $G_k \subset G^{\mathbb{Q}}_k$.

Theorem 3.10 (Milnor type theorem for rational Whitney towers)**.** *Suppose T is a twisted Whitney tower of order n in a rational homology 4-ball X which is bounded by an m-component link L in ∂X. Let $F \to \pi_1(X \smallsetminus T)$ be a homomorphism of the free group F generated by $x_1, \ldots, x_m$ which sends x_i to a meridian of the ith component of L. Then*

for each $k \leq n$, it induces an isomorphism

$$\frac{\pi_1(X \smallsetminus T)^{\mathbb{Q}}_{k+1}}{\pi_1(X \smallsetminus T)^{\mathbb{Q}}_{k+2}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \frac{F_{k+1}}{F_{k+2}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

To prove Theorem 3.10, we will use the following homology computation.

Lemma 3.11. *Suppose T is a twisted Whitney tower of order n in a rational homology 4-ball X. Then $\tilde{H}_i(X \smallsetminus T; \mathbb{Q}) \cong H^{3-i}(T, \partial T; \mathbb{Q})$, and the following hold.*

1. *The meridians of the order 0 disks form a basis for $H_1(X \smallsetminus T; \mathbb{Q})$.*
2. *$H_2(X \smallsetminus T; \mathbb{Q})$ is spanned by classes of tori which have standard basis curves α and β such that $\alpha \in \pi_1(X \smallsetminus T)_k$ and $\beta \in \pi_1(X \smallsetminus T)_{n-k+2}$ for some k.*

Proof. Let $G = \pi_1(X \smallsetminus T)$ and let N be a regular neighborhood of T in X. Let $\partial_- N := \partial N \cap \partial X$ and $\partial_+ N := \overline{\partial N \smallsetminus \partial_- N}$. Then

$$\begin{aligned} \tilde{H}_i(X \smallsetminus T; \mathbb{Q}) &\cong H_{i+1}(X, X \smallsetminus T; \mathbb{Q}) && \text{since } \tilde{H}_*(X; \mathbb{Q}) = 0, \\ &\cong H_{i+1}(N, \partial_+ N; \mathbb{Q}) && \text{by excision,} \\ &\cong H^{3-i}(N, \partial_- N; \mathbb{Q}) && \text{by duality for } (N, \partial_+ N, \partial_- N), \\ &\cong H^{3-i}(T, \partial T; \mathbb{Q}) && \text{since } (N, \partial_- N) \simeq (T, \partial T). \end{aligned} \tag{3.1}$$

Since $H_2(T, \partial T)$ is the free abelian group generated by the fundamental classes of the order zero disks rel boundary, the meridians of the order zero disks, which are dual to the fundamental classes, form a basis of $H_1(X \smallsetminus T; \mathbb{Q})$. This proves (1).

The remaining part is devoted to the proof of (2). Let m be the number of order zero disks. The pair $(T, \partial T)$ is homotopy equivalent to $K := \big(\bigsqcup_{i=1}^m (D^2, S^1)\big) \cup \big(\bigsqcup_j e_j\big)$, where each e_j is a 1-cell attached along a map $\partial e_j = S^0 \hookrightarrow \bigsqcup_{i=1}^m \operatorname{int} D^2$. Indeed each e_j is associated to either an unpaired intersection of T or a Whitney disk of order > 0. For each e_j, we will describe a torus C_j which is dual to e_j and has standard basis curves α and β such that $\alpha \in G_k$ and $\beta \in G_{n-k+2}$ for some k. Since the dual tori C_j span $H_2(X \smallsetminus T; \mathbb{Q})$ by (3.1), the conclusion (2) follows.

Case 1. Let e_j be a 1-cell of K associated to an unpaired intersection p between two disks D and D'. In T, e_j corresponds to an arc γ in T from D to D' through p. See Figure 2. Let C_j be the Clifford torus around p. The torus C_j is dual to the 1-cell e_j. A meridian μ of D and a meridian μ' of D' are standard basis curves of C_j. Let r and s are the orders of D and D' respectively. Then $\mu \in G_{r+1}$ and $\mu' \in G_{s+1}$ by Lemma 3.6. Since the intersection p is left unpaired, $r + s \geq n$. This shows that C_j satisfies the promised property.

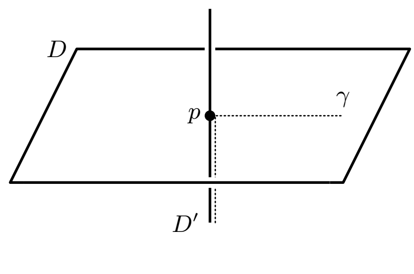

FIGURE 2. The arc γ passing through an unpaired intersection p.

Case 2. Let e_j be a 1-cell of K associated to a Whitney disk D'' between two disks D and D'. In T, e_j corresponds to an arc γ in T from D to D' through one of the involved intersections. See the $t=0$ picture in Figure 3. Let μ and μ' be meridians and r and s be the orders of D and D' respectively. Similarly to Case 1, we have $\mu \in G_{r+1}$ and $\mu' \in G_{s+1}$. In addition, since D'' has order $r+s+1$, $\partial D'' \in G_{n-r-s}$ by Lemma 3.8.

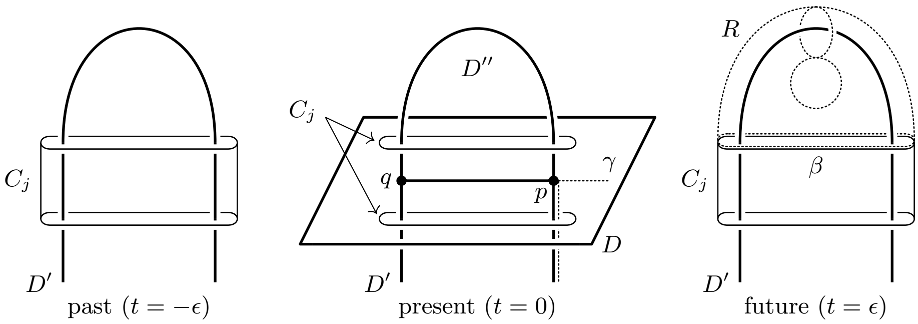

FIGURE 3. The torus C_j dual to the arc γ passing through a paired intersection p.

Let C_j be the torus illustrated in Figure 3; C_j is the union of two annuli in the $t=\pm\epsilon$ pictures, and additional two annuli connecting the boundary circles of the former annuli through $-\epsilon \le t \le \epsilon$; the connecting annuli are shown as two circles in the $t=0$ picture. It is straightforward to see that C_j is dual to the arc γ, similarly to the Clifford torus in Case 1.

Let β be the circle shown in the $t=\epsilon$ picture; β is the top boundary of the annulus part of C_j in the $t=\epsilon$ picture. The meridian μ of D and the circle β are standard basis curves of C_j. Since β is the boundary of the punctured torus R illustrated with dotted lines in the $t=\epsilon$ picture, and since μ' and $\partial D''$ are (homotopic to) standard basis curves of R, we have $\beta = [\mu', \partial D'']$. Since $\mu' \in G_{s+1}$ and $\partial D'' \in G_{n-r-s}$, we have $\beta \in G_{n-r+1}$. Since $\mu \in G_{r+1}$, this shows that the standard basis curves μ and β of the torus C_j satisfy the promised property. $\square$

Proof of Theorem 3.10. Let $G = \pi_1(X \smallsetminus T)$ where T is a twisted Whitney tower of order n in a rational homology 4-ball X bounded by an m-component link $L \subset \partial X$. By Lemma 3.11 (1), a given meridian map $F \to G$ induces an isomorphism $\mathbb{Q}^m \cong H_1(F;\mathbb{Q}) \cong H_1(G;\mathbb{Q})$.

By Lemma 3.11 (2), $H_2(X \smallsetminus T;\mathbb{Q})$ is generated by classes $[C]$ of tori C with standard basis curves α, β such that $\alpha \in G_k$ and $\beta \in G_{n-k+2}$ for some k. By a standard argument (e.g., see [FT95, (proofs of) Lemma 2.3 and Lemma 2.1]), such a toral class $[C]$ is contained in the kernel of $H_2(X \smallsetminus T) \to H_2(G/G_{n+1})$. Since $H_2(X \smallsetminus T;\mathbb{Q}) \to H_2(G;\mathbb{Q})$ is surjective, it follows that $H_2(G;\mathbb{Q}) \to H_2(G/G_{n+1};\mathbb{Q})$ is zero.

We now invoke Stallings-Dwyer theorem for rational coefficients [Sta65, Dwy75]: *if a group homomorphism $\Gamma \to G$ induces an isomorphism on $H_1(\Gamma;\mathbb{Q}) \cong H_1(G;\mathbb{Q})$ and an epimorphism*

$$H_2(\Gamma;\mathbb{Q}) \longrightarrow H_2(G;\mathbb{Q})/\operatorname{Ker}\{H_2(G;\mathbb{Q}) \to H_2(G/G_k;\mathbb{Q})\},$$

then it induces an isomorphism

$$(\Gamma^{\mathbb{Q}}_k/\Gamma^{\mathbb{Q}}_{k+1}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \xrightarrow{\cong} (G^{\mathbb{Q}}_k/G^{\mathbb{Q}}_{k+1}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}.$$

Applying this to the meridian map $F \to G$, we obtain an isomorphism

$$(F^{\mathbb{Q}}_{k+1}/F^{\mathbb{Q}}_{k+2}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong (G^{\mathbb{Q}}_{k+1}/G^{\mathbb{Q}}_{k+2}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

for each $k \le n$.

Therefore, to complete the proof of Theorem 3.10, it suffices to show that $F_k^{\mathbb{Q}} = F_k$ for all k. It is straightforward to verify this by an induction: $F_1^{\mathbb{Q}} = F = F_1$, and if $F_k^{\mathbb{Q}} = F_k$, then $F_k^{\mathbb{Q}}/[F, F_k^{\mathbb{Q}}] = F_k/F_{k+1} \cong \mathsf{L}_k$ is a finitely generated free abelian group (e.g., by the Hall basis theorem), and so by definition, we have

$$F_{k+1}^{\mathbb{Q}} = \operatorname{Ker}\{F_k \to (F_k/F_{k+1}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\} = \operatorname{Ker}\{F_k \to F_k/F_{k+1}\} = F_{k+1}. \qquad \square$$

3.4. Whitney towers and Milnor invariants

Now we are ready to prove the main result of this section.

Proof of Theorem 3.1. Suppose L is a framed link in S^3, X is a rational homology 4-ball with $\partial X = S^3$, and T is a twisted Whitney tower of order n in X bounded by L. We will prove that $\mu_k(L) = 0$ for $k < n$ and $\mu_n(L) = \eta_n(t_n^{\infty}(T))$.

Let $\pi = \pi_1(S^3 \smallsetminus L)$, $G = \pi_1(X \smallsetminus T)$, and let $\lambda_i \in \pi$ be a pushoff of the ith component of L taken along the given framing. (By Remark 2.7, λ_i is a zero linking longitude if $n > 0$.) By Lemma 3.9, the image of λ_i lies in G_{n+1}.

We proceed inductively. Suppose $k \le n$ and $\mu_{k-1}(L)$ has been shown to vanish. (We assume nothing for $k = 0$.) Let F be the free group of the same rank as the number of components of L. By Milnor's theorem [Mil57, Theorem 4] (see Section 3.1) and by Theorem 3.10, we obtain the following commutative diagram with vertical arrows isomorphisms:

$$\begin{tikzcd}
\pi_{k+1}/\pi_{k+2} \arrow[r] \arrow[d, "\cong"'] & G_{k+1}/G_{k+2} \arrow[r] & (G_{k+1}^{\mathbb{Q}}/G_{k+2}^{\mathbb{Q}}) \otimes \mathbb{Q} \arrow[d, "\cong"] \\
F_{k+1}/F_{k+2} \arrow[rr] & & (F_{k+1}/F_{k+2}) \otimes \mathbb{Q}
\end{tikzcd}$$

Let $w_i \in F_{k+1}/F_{k+2}$ be the image of λ_i. Then by definition, $\mu_k(L) = \sum_i X_i \otimes w_i \in \mathsf{L}_1 \otimes (F_{k+1}/F_{k+2}) \cong \mathsf{L}_1 \otimes \mathsf{L}_{k+1}$.

If $k \le n-1$, then since λ_i is sent into $G_{n+1} \subset G_{k+2}$, it follows that the image of λ_i in $(F_{k+1}/F_{k+2}) \otimes \mathbb{Q}$ is trivial. The bottom arrow of the diagram is a monomorphism since F_{k+1}/F_{k+2} is torsion free abelian. It follows that $w_i \in F_{k+1}/F_{k+2}$ is trivial. Therefore $\mu_k(L) = 0$.

If $k = n$, then the image of λ_i in G_{n+1}/G_{n+2} is given by Lemma 3.9. By comparing the longitude formula in Lemma 3.9 and the defining formula of η_n in Definition 2.8, it follows that

$$\mu_n(L) \otimes 1 = \eta_n(t_n^{\infty}(T)) \otimes 1 \in \mathsf{L}_1 \otimes (F_{n+1}/F_{n+2}) \otimes \mathbb{Q} = \mathsf{L}_1 \otimes \mathsf{L}_{n+1} \otimes \mathbb{Q}.$$

Since $\mathsf{L}_1 \otimes \mathsf{L}_{n+1} \to \mathsf{L}_1 \otimes \mathsf{L}_{n+1} \otimes \mathbb{Q}$ is injective, $\mu_n(L) = \eta_n(t_n^{\infty}(T))$ in $\mathsf{L}_1 \otimes \mathsf{L}_{n+1}$. $\square$

As a consequence of Theorem 3.1, we prove that Milnor invariants are preserved under rational Whitney tower concordance. It will be used in Section 5.

Corollary 3.12. *Suppose two framed links L and L' in S^3 are order $n+1$ twisted Whitney tower concordant in a rational homology $S^3 \times I$ ($n \ge 0$). If L bounds a twisted Whitney tower of order n in a rational homology 4-ball, then so does L', and furthermore $\mu_n(L) = \mu_n(L')$.*

Remark 3.13. In Section 4.2, we will show that a link L bounds a twisted Whitney tower of order n in a rational homology 4-ball if and only if $\mu_k(L) = 0$ for $k < n$. See Theorem 4.5.

Proof of Corollary 3.12. Let T be a twisted Whitney tower of order n bounded by L in a rational homology 4-ball, and let C be an order $n+1$ twisted Whitney tower concordance between L and L' in a rational homology $S^3 \times I$. Stacking T and C, we obtain an order $n+1$ twisted Whitney tower T' bounded by L' in another rational homology 4-ball. By Theorem 3.1, $\mu_k(L) = 0 = \mu_k(L')$ for $k < n$ and thus $\mu_n(L)$ and $\mu_n(L')$ are well-defined. Since all order n intersections of C are paired up by Whitney disks, we have $t_n^\infty(T) = t_n^\infty(T')$. By Theorem 3.1, it follows that $\mu_n(L) = \mu_n(L')$. $\square$

4. Links and Whitney towers in rational homology 4-space

In what follows we fix the number m of components of links. As in the introduction, define $\overline{\mathbb{W}}_n^\infty$ to be the set of framed m-component links L in S^3 bounding a twisted Whitney tower T of order n in a rational homology 4-ball with boundary S^3. (Recall that $\overline{\mathbb{W}}_0^\infty$ is the set of all links in S^3 by Remark 2.7.) Let $\overline{\mathsf{W}}_n^\infty$ to be the set of equivalence classes of links in $\overline{\mathbb{W}}_n^\infty$ under order $n+1$ twisted Whitney tower concordance in a rational homology $S^3 \times I$. (Readers may find that this is different from the defining condition in the introduction, but we will show that they are equivalent in Section 4.3, Corollary 4.7.)

In this section, we will show that Milnor invariants characterize links in $\overline{\mathsf{W}}_n^\infty$. Using this we will show that $\overline{\mathsf{W}}_n^\infty$ is an abelian group under band sum, and compute the structure of the abelian group $\overline{\mathsf{W}}_n^\infty$.

4.1. Some results of the integral twisted theory

Our approach relies in an essential way on the work of Conant, Schneiderman and Teichner on Whitney tower concordance in $S^3 \times I$ [CST12c, CST14, CST12a, CST12b]. In this subsection we quickly review parts of their work we need, focusing on the twisted case, and setup notations.

Let $\mathbb{W}_n^\infty$ be the set of m-component framed links in S^3 bounding a twisted Whitney tower of order n in D^4. Let W_n^∞ be the set of order $n+1$ twisted Whitney tower concordance classes of links in $\mathbb{W}_n^\infty$. Then the band sum of two classes is well defined on W_n^∞, independent of the choice of representative links and the choice of bands [CST12c, Lemma 3.4]. The set W_n^∞ is an abelian group under band sum. In particular, for $L, L' \in \mathbb{W}_n^\infty$, $[L] = [L']$ in W_n^∞ if and only if $L \#_\beta -L'$ is in $\mathbb{W}_{n+1}^\infty$ for some β. Often we write $\mathsf{W}_n^\infty = \mathbb{W}_n^\infty / \mathbb{W}_{n+1}^\infty$.

For a twisted Whitney tower T of order n, they define an invariant $\tau_n^\infty(T) \in \mathcal{T}_n^\infty$, which is the class of the formal sum $t_n^\infty(T)$ described in Definition 2.8, in a certain quotient $\mathcal{T}_n^\infty$ of the free abelian group generated by order n trees and order $\frac{n}{2}$ ∞-trees [CST12c]. We do not need the precise definition of $\mathcal{T}_n^\infty$. A key feature we will use is the following:

Theorem 4.1 (Order Raising [CST12c])**.** *If L bounds a twisted Whitney tower T of order n in D^4 with $\tau_n^\infty(T) = 0$ in $\mathcal{T}_n^\infty$, then L bounds a twisted Whitney tower of order $n+1$ in D^4.*

Any $\theta \in \mathcal{T}_n^\infty$ is realized by a link in the following sense: there is an epimorphism $R_n^\infty \colon \mathcal{T}_n^\infty \to \mathsf{W}_n^\infty$, called the *realization map*, such that $R_n^\infty(\theta)$ is the class of a link bounding an order n twisted Whitney tower T in D^4 with $\tau_n^\infty(T) = \theta$ [CST12c]. (This condition determines the class $R_n^\infty(\theta) \in \mathsf{W}_n^\infty$ uniquely by Theorem 4.1.)

The summation η_n described in Definition 3.5 induces a homomorphism $\eta_n \colon \mathcal{T}_n^\infty \to \mathsf{D}_n$ (e.g. see [CST14, Section 4.3]). Also, the Milnor invariant of order n gives rise to a homomorphism $\mu_n \colon \mathsf{W}_n^\infty \to \mathsf{D}_n$ [CST14]. We state some necessary facts as a theorem.

Theorem 4.2 (Conant-Schneiderman-Teichner [CST12c, CST14, CST12a, CST12b])**.**

(1) *For $n \not\equiv 2 \mod 4$, $\eta_n\colon \mathcal{T}^{\infty}_n \to \mathsf{D}_n$ and $\mu_n\colon \mathsf{W}^{\infty}_n \to \mathsf{D}_n$ are isomorphisms. For $n = 4k-2$, $\eta_{4k-2}\colon \mathcal{T}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2}$ is an epimorphism with kernel isomorphic to $\mathbb{Z}_2 \otimes \mathsf{L}_k$.*

(2) *For each θ in $\operatorname{Ker}\{\eta_{4k-2}\colon \mathcal{T}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2}\}$, $R^{\infty}_n(\theta) \in \mathsf{W}^{\infty}_{4k-2}$ is the class of a link obtained by starting with the figure eight knot, applying Bing doubling to certain components repeatedly, and then applying internal band sum operations connecting distinct components.*

The main conjecture is that R^{∞}_n is an isomorphism $\mathcal{T}^{\infty}_n \cong \mathsf{W}^{\infty}_n$ for $n \equiv 2 \mod 4$. This is equivalent to the higher order Arf invariant conjecture [CST12c].

4.2. Rational twisted Whitney tower filtration

In our characterization of links in $\overline{\mathbb{W}}^{\infty}_n$, the following straightforward observation based on earlier known facts is essential. Let $\mathsf{B}_{4k-2} := \operatorname{Ker}\{\eta_{4k-2}\colon \mathcal{T}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2}\}$. As stated in Theorem 4.2 (1), $\mathsf{B}_{4k-2} \cong \mathbb{Z}_2 \otimes \mathsf{L}_k$. We say that a link is *rationally slice* if it bounds slicing disks in a rational homology 4-ball. When R is a ring, a link is *R-slice* if it bounds slicing disks in an R-homology 4-ball.

Lemma 4.3. *For any $\theta \in \mathsf{B}_{4k-2}$, the realization $R^{\infty}_{4k-2}(\theta) \in \mathsf{W}^{\infty}_{4k-2}$ is represented by a $\mathbb{Z}[\frac{1}{2}]$-slice link $L(\theta) \in \mathbb{W}^{\infty}_{4k-2}$.*

Proof. The figure eight knot bounds a slice disk in a rational $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball, by [Cha07, Proof of Theorem 4.16, Figure 6]. If a link bounds slice disks in a 4-manifold, both Bing doubling operation on a component and internal band sum operation joining distinct components give another link bounding slice disks in the same 4-manifold. From this and Theorem 4.2 (2), the conclusion stated above follows. □

For brevity, we write $\mathsf{B}_n := \operatorname{Ker}\{\eta_n\colon \mathcal{T}^{\infty}_n \to \mathsf{D}_n\}$ for any n; for $n \not\equiv 2 \mod 4$, $\mathsf{B}_n = 0$ by Theorem 4.2 (1), and thus Lemma 4.3 is vacuously true.

For two m-component links L and L' in S^3, we denote by $L \#_\beta L'$ their band sum defined using a collection β of m bands joining the ith component of L and that of L' in the split union $L \sqcup L'$. That is, $L \#_\beta L'$ is the result of m internal band sum operations (ambient surgery) on $L \sqcup L'$.

We will often use that there is a standard genus zero cobordism in $S^3 \times I$ between $(L \sqcup L') \times 0 \subset S^3 \times 0$ and $(L \#_\beta L') \times 1 \subset S^3 \times 1$:

$$\big((L \sqcup L') \times [0, \tfrac{1}{2}]\big) \cup (\beta \times \tfrac{1}{2}) \cup \big((L \#_\beta L') \times [\tfrac{1}{2}, 1]\big).$$

Lemma 4.4. *Suppose L is a link bounding a twisted Whitney tower of order n in D^4. Then the following are equivalent:*

(1) *There is $\theta \in \mathsf{B}_n$ such that a band sum $L \#_\beta L(\theta)$ bounds a twisted Whitney tower of order $n+1$ in D^4 for any β. Here $L(\theta)$ is the link in Lemma 4.3.*
(2) *L bounds a twisted Whitney tower of order $n+1$ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball.*
(3) *L bounds a twisted Whitney tower of order $n+1$ in a rational homology 4-ball.*
(4) *$\mu_n(L) = 0$.*

Note that for $n \not\equiv 2 \mod 4$, (1) is equivalent to that L bounds a twisted Whitney tower of order $n+1$ in D^4.

Proof. Suppose $L \#_\beta L(\theta)$ bounds an order $n+1$ twisted Whitney tower T in D^4 as in (1). Then a standard argument gives an order $n+1$ twisted Whitney tower concordance in $S^3 \times I$, say T', between L and $L(\theta)$. Details are as follows: first attach to T a standard genus zero cobordism between $(L \#_\beta L(\theta)) \times 1$ and the split union $(L \sqcup L(\theta)) \times 0$ in

$S^3\times I$. This gives a tower T'' in D^4 bounded by $L\sqcup L(\theta)$. Identify D^4 with $\overline{S^3\smallsetminus D^3}\times I$ in such a way that L and $L(\theta)$ lie in the first and second summands of $\partial(\overline{S^3\smallsetminus D^3}\times I) = \overline{S^3\smallsetminus D^3}\cup_\partial -\overline{S^3\smallsetminus D^3} = S^3\,\#\,-S^3$ respectively. Then the promised $T'\subset S^3\times I$ is the image of T'' under the inclusion $\overline{S^3\smallsetminus D^3}\times I\subset S^3\times I$.

Attach to T' a slicing disk of $-L(\theta)$ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball, which exists by Lemma 4.3. The result is a twisted Whitney tower of order $n+1$ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball which is bounded by L. This shows (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (4) is an immediate consequence of Theorem 3.1.

Suppose (4) holds. Choose a twisted Whitney tower T of order n in D^4 bounded by L. Then $\eta_n(\tau^\infty_n(T)) = \mu_n(L) = 0$ by using Theorem 3.1 (or the original integral version [CST14, Theorem 6]). Therefore $\tau^\infty_n(T)\in \mathsf{B}_{4k-2}$. Let $\theta := -\tau^\infty_n(T)$. Then $L(\theta)$ bounds a twisted Whitney tower T'' in D^4 with $\tau^\infty_n(T'') = -\tau^\infty_n(T)$. Attach the disjoint union of T and T'' to a standard genus zero cobordism between $L\,\#_\beta\, L(\theta)$ and $L\sqcup L(\theta)$, to obtain an order n twisted Whitney tower with $\tau^\infty_n = \tau^\infty_n(T)+\tau^\infty_n(T'') = 0$. By Theorem 4.1, it follows that $L\,\#_\beta\, L(\theta)$ lies in $\mathbb{W}^\infty_{n+1}$. This shows (4) $\Rightarrow$ (1). ☐

We will use connected sum of links as a special case of band sum. A precise description is as follows. Let L be a link with m components in S^3. Fix m distinct interior points $z_1,\dots,z_m\in D^2$. Choose an embedding $b\colon D^2\times I\to S^3$ such that the inverse image of the ith component of L under b is equal, as an oriented arc, to $z_i\times I$. We call b a *basing* for L. Let L' be another m-component link with a basing b'. Let $Y=\overline{S^3\smallsetminus b(D^2\times I)}$ and $Y'=\overline{S^3\smallsetminus b'(D^2\times I)}$. Define the *connected sum* $L\,\#_{(b,b')}\,L'$ by $(S^3, L\,\#_{(b,b')}\,L') = (Y, L\cap Y)\cup_\partial (Y', L\cap Y')$ where ∂Y is identified with $\partial Y'$ under $b(z,t)\mapsto b'(z,1-t)$, $(z,t)\in\partial(D^2\times I)$. That is, the connected sum $L\,\#_{(b,b')}\,L'$ is the band sum defined using the pair of basings (b,b') as bands.

Now we are ready to present a complete characterization of links bounding a twisted Whitney tower of a given order in a rational and $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball.

Theorem 4.5. *For any link L in S^3 and $n\geq 0$, the following are equivalent:*

1. *$L\in\overline{\mathbb{W}}^\infty_{n+1}$, that is, L bounds a twisted Whitney tower of order $n+1$ in a rational homology 4-ball.*
2. *$\mu_k(L)=0$ for $k\leq n$.*
3. *For any basing b for L, there is a rationally slice link L_0 with a basing b_0 such that $L\,\#_{(b,b_0)}\,L_0\in\mathbb{W}^\infty_{n+1}$.*
4. *For any basing b for L, there is a $\mathbb{Z}[\frac{1}{2}]$-slice link L_0 with a basing b_0 such that $L\,\#_{(b,b_0)}\,L_0\in\mathbb{W}^\infty_{n+1}$.*
5. *L bounds a twisted Whitney tower of order $n+1$ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball.*

From Theorem 4.5, the twisted case of Theorem B in the introduction follows immediately: *for any subring R of $\mathbb{Q}$ containing $\frac{1}{2}$, a link in S^3 bounds a twisted Whitney tower of order n in an R-homology 4-ball if and only if the link bounds a twisted Whitney tower of order n in a rational homology 4-ball.*

Also, Theorem C in the introduction is exactly the equivalence of (1) and (2) in Theorem 4.5.

Proof of Theorem 4.5. (1) $\Rightarrow$ (2) is Theorem 3.1 (1). Suppose (2) holds. Since $L\in\mathbb{W}^\infty_0$ and $\mu_0(L)=0$, there is $\theta_0\in\mathsf{B}_0$ and a basing b_0 for $L(\theta_0)$ such that $L\,\#_{(b,b_0)}\,L(\theta_0)\in\mathbb{W}^\infty_1$ for any b for L, by Lemma 4.4. Choose a basing b_0' for $L(\theta_0)$ which is disjoint from b_0. If $n\geq 1$, then by the same argument, using $\mu_1(L)=0$, there is $\theta_1\in\mathsf{B}_1$ and two disjoint basings b_1 and b_1' for $L(\theta_1)$ such that $(L\,\#_{(b,b_0)}\,L(\theta_0))\,\#_{(b_0',b_1)}\,L(\theta_1)\in\mathbb{W}^\infty_2$. Repeating

this, choose $\theta_i \in \mathsf{B}_i$ and disjoint basings b_i and b_i' for $L(\theta_i)$ for $i = 0, \dots, n$ such that

$$\Big(\cdots \big(L \mathop{\#}_{(b,b_0)} L(\theta_0) \big) \mathop{\#}_{(b_0',b_1)} \cdots \Big) \mathop{\#}_{(b_{n-1}',b_n)} L(\theta_n) \in \mathbb{W}^{\infty}_{n+1}.$$

Since the basings are disjoint, the above connected sum operations are associative. It follows that $L_0 := L(\theta_0) \#_{(b_0',b_1)} \cdots \#_{(b_{n-1}',b_n)} L(\theta_n)$ with the basing b_0 satisfies (4). This shows (2) $\Rightarrow$ (4). (4) $\Rightarrow$ (3) is straightforward.

Both (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are are shown by the standard argument used in the proof of (1) $\Rightarrow$ (2) of Lemma 4.4, using that L_0 is rationally slice and $\mathbb{Z}[\frac{1}{2}]$-slice respectively. (5) $\Rightarrow$ (1) is trivial. The completes the proof. □

4.3. Rational twisted graded quotient

For brevity, write $L \sim L'$ if two framed links L and L' in $\overline{\mathbb{W}}^{\infty}_n$ are order $n+1$ twisted Whitney tower concordant in a rational homology $S^3 \times I$. Recall that $\overline{\mathsf{W}}^{\infty}_n = \overline{\mathbb{W}}^{\infty}_n / \sim$.

Theorem 4.6.

(1) $\overline{\mathsf{W}}^{\infty}_n$ *is an abelian group under band sum* $[L] + [L'] = [L \#_\beta L']$.

(2) $\mu_n \colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ *is a group isomorphism.*

(3) *For the m-component case,* $\overline{\mathsf{W}}^{\infty}_n$ *is a free abelian group of rank* $\mathcal{M}(m,n)$*, where* $\mathcal{M}(m,k)$ *is the number defined in Remark 3.2.*

(4) $\mathsf{W}^{\infty}_n \to \overline{\mathsf{W}}^{\infty}_n$ *is an epimorphism with kernel equal to* $\mathsf{K}^{\infty}_n := \mathrm{Ker}\{\mu_n \colon \mathsf{W}^{\infty}_n \to \mathsf{D}_n\}$. $\mathsf{W}^{\infty}_n \cong \overline{\mathsf{W}}^{\infty}_n$ *for* $n \not\equiv 2 \mod 4$.

In the following proof, we will use Krushkal's additivity [Kru98]: *if* $\mu_q(L) = 0 = \mu_q(L')$ *for* $q < n$*, then* $\mu_n(L \#_\beta L') = \mu_n(L) + \mu_n(L')$ *for any bands* β. It can also be seen by using Theorem 3.1.

Proof of Theorem 4.6. We first claim that for $L, L' \in \overline{\mathbb{W}}^{\infty}_n$, $L \sim L'$ if and only if $\mu_n(L) = \mu_n(L')$. The only if direction is true by Corollary 3.12. Conversely, if $\mu_n(L) = \mu_n(L')$, then for any choice of bands β, $\mu(L \#_\beta -L') = \mu(L) - \mu(L') = 0$ by the additivity. By Theorem 4.5, it follows that $L \#_\beta -L' \in \overline{\mathbb{W}}^{\infty}_{n+1}$. It implies $L \sim L'$ by the standard argument for band sum which was used in the proof of (1) $\Rightarrow$ (2) of Lemma 4.4. This completes the proof of the claim.

By the claim, $\mu_n \colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ is an injective *function.* Since the diagram

$$\begin{tikzcd} \mathsf{W}^{\infty}_n \arrow[rr] \arrow[dr, "\mu_n"'] & & \overline{\mathsf{W}}^{\infty}_n \arrow[dl, "\mu_n"] \\ & \mathsf{D}_n & \end{tikzcd} \tag{4.1}$$

is commutative and since $\mu_n \colon \mathsf{W}^{\infty}_n \to \mathsf{D}_n$ is surjective by Theorem 4.2 (1), it follows that $\mu_n \colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ is surjective. Therefore $\mu_n \colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ is bijective.

Also, from the claim, it follows that the class $[L \#_\beta L']$ of a band sum is determined by the classes $[L]$ and $[L] \in \overline{\mathsf{W}}^{\infty}_n$, independent of the choice of β, since $\mu_n(L \#_\beta L') = \mu_n(L) + \mu_n(L')$ is determined by $\mu_n(L)$ and $\mu_n(L')$. Thus $[L] + [L'] = [L \#_\beta L']$ is a well defined operation on $\overline{\mathsf{W}}^{\infty}_n$.

Since D_n is a group and $\mu_n \colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ is a bijective function preserving the addition, $\overline{\mathsf{W}}^{\infty}_n$ is a group under the addition and $\mu_n \colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ is a group isomorphism. This proves (1) and (2).

Since D_n is a free abelian group of rank $\mathcal{M}(m,n)$ (see Remark 3.2), so is $\overline{\mathsf{W}}^{\infty}_n$. This shows (3).

Since $\mu_n\colon \overline{\mathsf{W}}^{\infty}_n \to \mathsf{D}_n$ is an isomorphism, from (4.1) it follows that $\mathsf{W}^{\infty}_n \to \overline{\mathsf{W}}^{\infty}_n$ is surjective and has kernel $\mathsf{K}^{\infty}_n := \operatorname{Ker}\{\mu_n\colon \mathsf{W}^{\infty}_n \to \mathsf{D}_n\}$. By Theorem 4.2 (1), $\mathsf{K}^{\infty}_n = 0$ for $n \not\equiv 2 \mod 4$. This shows (4). □

Since $[L] = 0$ in $\overline{\mathsf{W}}^{\infty}_n$ if and only if $L \in \overline{\mathbb{W}}^{\infty}_{n+1}$, the following is a direct consequence of Theorem 4.6 (1).

Corollary 4.7. $[L] = [L']$ *in* $\overline{\mathsf{W}}^{\infty}_n$ *if and only if* $L \#_\beta -L' \in \overline{\mathbb{W}}^{\infty}_{n+1}$

We conclude this section with a discussion on the higher order Arf invariants. Recall that $\mathsf{B}_{4k-2} = \operatorname{Ker}\{\eta_{4k-2}\colon \mathcal{T}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2}\} \cong \mathbb{Z}_2 \otimes \mathsf{L}_k$ and $\mathsf{K}^{\infty}_{4k-2} = \operatorname{Ker}\{\mu_{4k-2}\colon \mathsf{W}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2}\}$. In [CST12c], Conant, Schneiderman and Teichner showed that $R^{\infty}_{4k-2}\colon \mathcal{T}^{\infty}_{4k-2} \to \mathsf{W}^{\infty}_{4k-2}$ restricts to an epimorphism $\alpha^{\infty}_k\colon \mathsf{B}_{4k-2} \twoheadrightarrow \mathsf{K}^{\infty}_{4k-2}$. They defined the *kth higher order Arf invariant* by

$$\operatorname{Arf}_k := (\alpha^{\infty}_k)^{-1}\colon \mathsf{K}^{\infty}_{4k-2} \xrightarrow{\cong} \mathsf{B}_{4k-2}/\operatorname{Ker}\alpha_k.$$

The higher order Arf invariant conjecture asserts that α^{∞}_k is an isomorphism. In particular, it claims that Arf_k is not identically trivial.

Using the definition of Arf_k, it is straightforward to reformulate Theorem 4.6 (4) to the following statement:

Corollary 4.8. *The epimorphism* $\mathsf{W}^{\infty}_n \to \overline{\mathsf{W}}^{\infty}_n$ *is an isomorphism if and only if either* $n \not\equiv 2 \mod 4$, *or* $n = 4k-2$ *and* $\operatorname{Arf}_k \equiv 0$.

Theorem D in the introduction is an immediate consequence of Corollary 4.8.

5. Framed classification

Let $\overline{\mathbb{W}}_n$ be the set of framed links in S^3 which bound an order n framed Whitney tower in a rational homology 4-ball. The goal of this section is to understand the structure of the filtration $\{\overline{\mathbb{W}}_n\}$ and its graded quotients $\overline{\mathsf{W}}_n$ which is a framed analog of $\overline{\mathsf{W}}^{\infty}_n$. We will define $\overline{\mathsf{W}}_n$ precisely in Section 5.3. The main result is as follows.

Theorem 5.1. *For the m-component case, the following hold.*

(1) *The Milnor invariant of order n gives rise to an epimorphism* $\mu_n\colon \overline{\mathsf{W}}_n \to \mathbb{Z}^{\mathcal{M}(m,n)}$ *onto a free abelian group of rank* $\mathcal{M}(m,n)$.
(2) *If n is even, then* μ_n *is an isomorphism* $\overline{\mathsf{W}}_n \cong \mathbb{Z}^{\mathcal{M}(m,n)}$.
(3) *If* $n = 2\ell - 1$, *there is a short exact sequence*

$$0 \longrightarrow (\mathbb{Z}_2)^{\mathcal{R}(m,\ell+1)} \longrightarrow \overline{\mathsf{W}}_n \xrightarrow{\mu_n} \mathbb{Z}^{\mathcal{M}(m,n)} \longrightarrow 0$$

where $\operatorname{Ker}\{\mu_n\}$ *is identified with* $(\mathbb{Z}_2)^{\mathcal{R}(m,\ell+1)} \cong \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}$ *via the higher order Sato-Levine invariant* $\operatorname{SL}_{2\ell-1}$. *Consequently,* $\mathsf{W}_n \cong \mathbb{Z}^{\mathcal{M}(m,n)} \oplus (\mathbb{Z}_2)^{\mathcal{R}(m,\ell+1)}$.

The *higher-order Sato-Levine invariant* SL_{2k-1} which appears in Theorem 5.9 (3) is essential in this section. Here we describe its definition following [CST12c]. Recall that D_n is the kernel of the bracket map $\mathsf{L}_1 \otimes \mathsf{L}_{n+1} \to \mathsf{L}_{n+2}$ given by $X_i \otimes Y \mapsto [X_i, Y]$. Suppose $n = 2k$. Due to Levine [Lev02, Theorem 1 and Corollary 2.2], the quotient of D_{2k} modulo the subgroup generated by $\{\eta_{2k}(t) \mid t \text{ is an order } 2k \text{ tree}\}$ is isomorphic to $\mathbb{Z}_2 \otimes L_{k+1}$. Let $\operatorname{sl}_{2k}\colon \mathsf{D}_{2k} \twoheadrightarrow \mathbb{Z}_2 \otimes L_{k+1}$ be the quotient map.

Definition 5.2 (Higher order Sato-Levine invariant)**.** For a link $L \subset S^3$ with $\mu_i(L) = 0$ for $i \leq 2k-1$, $\operatorname{SL}_{2k-1}(L) := \operatorname{sl}_{2k}(\mu_{2k}(L))$.

In Section 5.1, we will review some necessary results on framed Whitney towers in D^4, from the work of Conant, Schneiderman and Teichner. In Section 5.2, we will present a complete characterization of links in $\overline{\mathbb{W}}_n$ in terms of the Milnor invariant and higher order Sato-Levine invariants (see Theorem 5.6), and finally in Section 5.3, we will define the graded quotient $\overline{\mathsf{W}}_n$ and compute its structure to prove Theorem 5.1.

5.1. Some results from the integral framed theory

All results discussed in this subsection are from the work Conant, Schneiderman and Teichner [CST12c, CST14, CST12a, CST12b]. Similarly to the twisted case, let $\mathbb{W}_n$ be the set of m-component framed links in S^3 bounding an order n framed Whitney tower in D^4. Order $n+1$ framed Whitney tower concordance in D^4 is an equivalence relation on $\mathbb{W}_n$. Let W_n be the set of equivalence classes. The band sum of two classes is well defined on W_n (particularly independent of the choice of bands) [CST12c, Lemma 3.4], and W_n is an abelian group under band sum. Two links $L, L' \in \mathbb{W}_n$ represent the same element in W_n if and only if $L \#_\beta -L' \in \mathbb{W}_{n+1}$ for some β. Often we write $\mathsf{W}_n = \mathbb{W}_n/\mathbb{W}_{n+1}$.

In the study of the framed theory, they use a framed analog $\widetilde{\mathcal{T}}_n$ of the group $\mathcal{T}^{\infty}_n$ discussed in Section 4.1. The group $\widetilde{\mathcal{T}}_n$ is a quotient of the free abelian group generated by order n trees (without using ∞-trees), modulo certain relations. We omit its precise definition since we will use only the results discussed below. For an order n framed Whitney tower T, the formal sum $t^{\infty}_n(T)$ described in Definition 2.8 does not have any ∞-tree summand and thus represents an element $\tilde\tau_n(T) \in \widetilde{\mathcal{T}}_n$. Conversely, there is an epimorphism $\widetilde{R}_n \colon \widetilde{\mathcal{T}}_n \to \mathsf{W}_n$, called the *realization map*, such that $\widetilde{R}_n(\phi)$ is the class of a link bounding an order n framed Whitney tower T in D^4 with $\tilde\tau_n(T) = \phi$.

Theorem 5.3 (Framed Order Raising [CST12c, Theorem 4.4])**.** *If a link L bounds an order n framed Whitney tower T in D^4 with $\tilde\tau_n(T) = 0 \in \widetilde{\mathcal{T}}_n$, then L bounds an order $n+1$ framed Whitney tower in D^4.*

For even n, they showed that $\widetilde{\mathcal{T}}_{2\ell} \cong \mathbb{Z}^{\mathcal{M}(m,n)}$ where m is the number of link components, using their proof of the Levine conjecture [CST12a, CST12c]. (Recall that $\mathcal{M}(m,n)$ is the rank of D_n; see Remark 3.2.) In fact, there is a homomorphism $\widetilde{\mathcal{T}}_n \to \mathcal{T}^{\infty}_n$ taking the class of an order n tree to the class of the same tree, and for $n = 2\ell$, the composition $\widetilde{\mathcal{T}}_{2\ell} \to \mathcal{T}^{\infty}_{2\ell} \xrightarrow{\eta_{2\ell}} \mathsf{D}_{2\ell}$ is a monomorphism whose image has the same rank as $\mathsf{D}_{2\ell}$.

For odd $n = 2\ell - 1$, the structure of $\widetilde{\mathcal{T}}_{2\ell-1}$ is more involved, as described below. The boundary twist operation defined in [FQ90, Section 1.3] changes a twisted Whitney disk to a framed Whitney disk at the cost of introducing new intersections. Using this (together with IHX), in [CST12c], it was observed that a twisted Whitney tower T of order 2ℓ can be changed to a framed Whitney tower of order $2\ell-1$, which we denote by $\partial^{\infty}(T)$. In terms of the associated trees, this geometric modification changes an order ℓ ∞-tree of the form $\infty \prec^{J}_{i}$ to the order $2\ell-1$ tree $i \prec^{J}_{J}$. (Here i denotes a univariant vertex and J is a subtree; any ∞-tree can be changed to the form of $\infty \prec^{J}_{i}$ by IHX.) This gives rise to a homomorphism $\partial^{\infty} \colon \mathcal{T}^{\infty}_{2\ell} \to \widetilde{\mathcal{T}}_{2\ell-1}$ satisfying $\tilde\tau_{2\ell-1}(\partial^{\infty}(T)) = \partial^{\infty}(\tau^{\infty}_{2\ell}(T))$ for a twisted Whitney tower T of order 2ℓ. The following commutative diagram, which we discuss below, computes the structure of $\widetilde{\mathcal{T}}_{2\ell-1}$ [CST12c, CST14, CST12a, CST12b].

(5.1)
$$\begin{tikzcd}
\mathsf{B}_{2\ell} \arrow[r, "\partial^{\infty}|", "\cong"'] \arrow[d, tail] & \mathsf{B}^{\mathrm{SL}}_{2\ell-1} \arrow[dd, tail] & \cong \begin{cases} \mathbb{Z}_2 \otimes \mathsf{L}_k & \text{if } \ell = 2k-1 \\ 0 & \text{if } \ell = 2k \end{cases} & \\
\mathcal{T}^{\infty}_{2\ell} \arrow[rr, "\partial^{\infty}"] \arrow[dr, two heads] \arrow[ddd, two heads, "\eta_{2\ell}"] & & \widetilde{\mathcal{T}}_{2\ell-1} \arrow[r, two heads] & \mathcal{T}^{\infty}_{2\ell-1} \arrow[d, "\cong"', "\eta_{2\ell-1}"] \\
& \mathbb{Z}_2 \otimes \mathsf{L}'_{\ell+1} \arrow[ur, tail] \arrow[dd, two heads] & & \mathsf{D}_{2\ell-1} \\
& & & \\
\mathsf{D}_{2\ell} \arrow[r, two heads, "\mathrm{sl}_{2\ell}"'] & \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1} & &
\end{tikzcd}$$

(1) The row starting with $\mathcal{T}^{\infty}_{2\ell}$ is exact. By Theorem 4.2 (1), $\mathcal{T}^{\infty}_{2\ell-1} \cong \mathsf{D}_{2\ell-1}$ under $\eta_{2\ell-1}$.

(2) The image of $\partial^{\infty}\colon \mathcal{T}^{\infty}_{2\ell} \to \widetilde{\mathcal{T}}_{2\ell-1}$ is isomorphic to $\mathbb{Z}_2 \otimes \mathsf{L}'_{\ell+1}$, where $\mathsf{L}'_{\ell+1}$ is the degree $\ell+1$ part of Levine's *quasi-Lie algebra* [Lev02]. The abelian group $\mathsf{L}'_{\ell+1}$ is defined by replacing the alternativity relation $[X,X] = 0$ of $\mathsf{L}_{\ell+1}$ with the antisymmetry relation $[X,Y] + [Y,X] = 0$. Regarding $\mathsf{L}'_{\ell+1}$, we will need only (3) and (4) below.

(3) There is a homomorphism $\mathbb{Z}_2 \otimes \mathsf{L}'_{\ell+1} \to \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}$ taking the class of an $(\ell+1)$-fold bracket in $\mathsf{L}'_{\ell+1}$ to the class of the same bracket in $\mathsf{L}_{\ell+1}$. It is an epimorphism fitting into the bottom left square.

(4) Let $\mathsf{B}^{\mathrm{SL}}_{2\ell-1} := \mathrm{Ker}\{\mathbb{Z}_2 \otimes \mathsf{L}'_{\ell+1} \to \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}\}$. Then ∂^{∞} restricts to an isomorphsm $\mathsf{B}_{2\ell} \cong \mathsf{B}^{\mathrm{SL}}_{2\ell-1}$. Recall that B_n is the kernel of $\eta_n\colon \mathcal{T}^{\infty}_n \to \mathsf{D}_n$, and isomorphic to $\mathbb{Z}_2 \otimes \mathsf{L}_k$ if $n = 4k-2$, and 0 if $n \not\equiv 2 \mod 4$, as discussed in Section 4.1.

5.2. Framed rational Whitney tower filtration

Lemma 5.4. *For any $\phi \in \mathsf{B}^{\mathrm{SL}}_{4k-3} \cong \mathbb{Z}_2 \otimes \mathsf{L}_k$, there is a rationally slice link $L(\phi)$ in S^3 which bounds a framed Whitney tower T of order $4k-3$ in D^4 with $\tilde\tau_{4k-3}(T) = \phi$.*

Proof. By Lemma 4.3, there is a rationally slice link L which bounds a twisted Whitney tower T' of order $4k-2$ in D^4 with $\tau^{\infty}_{4k-2}(T') = (\partial^{\infty})^{-1}(\phi)$. Then $T := \partial^{\infty}(T')$ is an order $4k-3$ framed tower in D^4 bounded by L, and $\tilde\tau_{4k-3}(T) = \partial^{\infty}(\tau^{\infty}_{4k-2}(T')) = \phi$. So $L(\phi) := L$ satisfies the desired properties. □

Similarly to the convention for B_n in the twisted case, let $\mathsf{B}^{\mathrm{SL}}_{2\ell} = 0$ for brevity. Then Lemma 5.4 holds for any order n as well as $n = 4k-3$.

The following is a framed case analog of Lemma 4.4.

Lemma 5.5. *Suppose L is a link bounding a framed order n Whitney tower in D^4. Then the following are equivalent:*

(1) *There is $\phi \in \mathsf{B}^{\mathrm{SL}}_n$ such that any band sum $L \#_\beta L(\phi)$ bounds a framed order $n+1$ Whitney tower in D^4.*

(2) *L bounds a framed Whitney tower of order $n+1$ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball.*

(3) *L bounds a framed Whitney tower of order $n+1$ in a rational homology 4-ball.*

(4) *$\mu_n(L) = 0$, and in addition when $n = 2\ell-1$, $\mathrm{SL}_{2\ell-1}(L) = 0$.*

Proof. (1) ⇒ (2) is proven in the exactly same way as (1) ⇒ (2) of Lemma 4.3, using that $L(\phi)$ is rationally slice. (2) ⇒ (3) is trivial.

Suppose (3) holds, that is, there is an order $n+1$ framed Whitney tower T in a rational homology 4-ball bounded by L. Since T is an order $n+1$ twisted Whitney tower, $\mu_n(L) = 0$ by Theorem 3.1. If $n = 2\ell-1$, then $\mathrm{SL}_{2\ell-1}(L) = \mathrm{sl}_{2\ell}(\mu_{2\ell}(L)) =$

$\mathrm{sl}_{2\ell}(\eta_{2\ell}(t^{\infty}_{2\ell}(T)))$ by Definition 5.2 and Theorem 3.1. Since T is framed, $t^{\infty}_{2\ell}(T)$ has no ∞-tree summand, that is, all the summands are order 2ℓ trees. By the definition of $\mathrm{sl}_{2\ell}$, it follows that $\mathrm{sl}_{2\ell}(\eta_{2\ell}(t^{\infty}_{2\ell}(T))) = 0$. This shows that (4) holds.

Suppose (4) holds. If $n = 2\ell$, then for any fixed order n framed Whitney tower T in D^4 bounded by L, $\eta_{2\ell}(\tilde\tau_{2\ell}(T)) = \mu_{2\ell}(L) = 0$. Since $\widetilde{\mathcal{T}}_{2\ell} \to \mathcal{T}^{\infty}_{2\ell} \xrightarrow{\eta_{2\ell}} D_{2\ell}$ is injective, $\tilde\tau_{2\ell}(T) = 0$ in $\widetilde{\mathcal{T}}_{2\ell}$. By Theorem 5.3, L bounds a framed order $n+1$ Whitney tower in D^4. In particular, (1) holds (with $\phi = 0$). If $n = 2\ell - 1$, then since $\mu_{2\ell-1}(L) = 0$ and $\mu_{2\ell-1}\colon \mathsf{W}^{\infty}_{2\ell-1} \to D_{2\ell-1}$ is an isomorphism by Theorem 4.2 (1), L bounds an order 2ℓ twisted Whitney tower T in D^4. Using the hypothesis and Theorem 3.1, we obtain

$$0 = \mathrm{SL}_{2\ell-1}(L) = \mathrm{sl}_{2\ell}(\mu_{2\ell}(L)) = \mathrm{sl}_{2\ell}(\eta_{2\ell}(\tau^{\infty}_{2\ell}(T))).$$

It follows that $\tilde\tau_{2\ell-1}(\partial^{\infty}(T)) = \partial^{\infty}(\tau^{\infty}_{2\ell}(T)) \in \mathsf{B}^{\mathrm{SL}}_{2\ell-1}$, using the diagram (5.1). Let $\phi = -\tilde\tau_{2\ell-1}(\partial^{\infty}(T))$. Then any band sum $L \#_\beta L(\phi)$ bounds a framed Whitney tower T' with $\tilde\tau_{2\ell-1}(T') = \tilde\tau_{2\ell-1}(\partial^{\infty}(T)) + \phi = 0 \in \widetilde{\mathcal{T}}_{2\ell-1}$. By Theorem 5.3, $L \#_\beta L(\phi)$ bounds a framed order $n+1$ Whitney tower in D^4. This completes the proof of (4) $\Rightarrow$ (1). ☐

Once Lemma 5.4 is given, the following theorem is proven by the argument of the proof of its twisted analog Theorem 4.5, using Lemma 5.4 in place of Lemma 4.3.

Theorem 5.6. *For a link L in S^3 and $n \geq 0$, the following are equivalent:*

(1) *$L \in \overline{\mathbb{W}}_{n+1}$, that is, L bounds a framed Whitney tower of order $n+1$ in a rational homology 4-ball.*
(2) *$\mu_k(L) = 0$ for $k \leq n$, and in addition when $n = 2\ell - 1$, $\mathrm{SL}_{2\ell-1}(L) = 0$.*
(3) *For any basing b for L, there is a rationally slice link L_0 with a basing b_0 such that $L \#_{(b,b_0)} L_0 \in \mathbb{W}_{n+1}$.*
(4) *For any basing b for L, there is a $\mathbb{Z}[\frac{1}{2}]$-slice link L_0 with a basing b_0 such that $L \#_{(b,b_0)} L_0 \in \mathbb{W}_{n+1}$.*
(5) *L bounds a twisted Whitney tower of order $n+1$ in a $\mathbb{Z}[\frac{1}{2}]$-homology 4-ball.*

We omit details of the proof.

5.3. Group structure on the rational framed graded quotients

In this subsection we will formulate the "graded quotient" $\overline{\mathsf{W}}_n$ of the rational framed filtration $\{\overline{\mathbb{W}}_n\}$ and compute its structure. Rather unexpectedly, the main remaining difficulty is to show that the graded quotient has a group structure under band sum. Once this is resolved, the group can be computed via Milnor invariants and higher order Sato-Levine invariants, using Theorem 5.6. To estabilish a group structure, it appears to have significant advantage to adapt the following definition of an equivalence relation, instead of framed Whitney tower concordance.

Definition 5.7. On the set $\overline{\mathbb{W}}_n$, define a relation $\approx$ by $L \approx L'$ if $L \#_\beta -L' \in \overline{\mathbb{W}}_{n+1}$ for some β.

Lemma 5.8. *On $\overline{\mathbb{W}}_n$, $\approx$ is an equivalence relation.*

It is straightforward to verify that $\approx$ is symmetric and reflexive. In the proof of transitivity, we use the following two facts: (i) a link in $\overline{\mathbb{W}}_n$ can always be represented by a link in $\mathbb{W}_n$, due to Theorem 5.6, and (ii) band sum is well-defined on $\mathsf{W}_n = \mathbb{W}_n/\mathbb{W}_{n+1}$, due to [CST12c].

Proof of Lemma 5.8. We will prove transitivity. Suppose L, L' and L'' are in $\overline{\mathbb{W}}_n$ and $L \#_\beta -L'$, $L' \#_\gamma -L''$ are in $\overline{\mathbb{W}}_{n+1}$. We need to show that $L \#_\alpha -L''$ is in $\overline{\mathbb{W}}_{n+1}$ for some choice of bands α.

In what follows, we will repeatedly use a standard fact that if L_0 is rationally slice, then for any link L and for any β, $L \#_\beta L_0$ is rationally concordant to L. The proof is straightforward: choose slice disks Δ for L_0 in a rational homology $S^3 \times I$, choose an arc in the rational homology $S^3 \times I$ which joins two boundary components and which is disjoint from Δ, and replace a tubular neighborhood of the arc with $(S^3 \smallsetminus (\text{3-ball disjoint from } L), L) \times I$ to obtain a cobordism between $L \sqcup L_0$ and L. Attach to this a standard genus zero cobordism in $S^3 \times I$ between $L \#_\beta L_0$ and $L \sqcup L_0$ to obtain a concordance between $L \#_\beta L_0$ to L in a rational homology $S^3 \times I$. The same argument shows that if $L_0 \in \overline{\mathbb{W}}_n$, then $L \#_\beta L_0$ is order n framed Whitney tower concordant to L in a rational homology $S^3 \times I$.

Begin with the split union $L \sqcup -L' \sqcup L' \sqcup -L''$, and regard β and γ as disjoint bands joining components of sublinks of this split union. Choose a collection of bands δ disjoint from β and γ to define a band sum $-L' \#_\delta L'$ of the sublinks $-L'$ and L'. Then

$$J := (L \underset{\beta}{\#} -L') \underset{\delta}{\#} (L' \underset{\gamma}{\#} -L'')$$

is defined. The link J bounds a framed Whitney tower of order $n+1$ in a rational homology 4-ball, since so do $L \#_\beta -L'$ and $L' \#_\gamma -L''$. Fix arbitrarily given bands α on $L \sqcup -L''$ to define a band sum $L \#_\alpha -L''$. We claim that J is order $n+1$ framed Whitney tower concordant to $L \#_\alpha -L''$ in some rational homology $S^3 \times I$. Stacking the claimed Whitney tower concordance with the above Whitney tower bounded by J, it follows that $L \#_\alpha -L''$ bounds a framed Whitney tower of order $n+1$ in a rational homology 4-ball. This completes the proof.

The remaining part is devoted to the proof of the claim. For a basing c of a link R, the mirror image of c with reversed orientation is a basing of $-R$. Denote this basing by $-c$. Any basing of $-R$ is of the form of $-c$. Choose basings b, b' and $-b''$ for the sublinks L, L' and $-L''$ of the split union $L \sqcup -L' \sqcup L' \sqcup -L''$ respectively. We may assume that b, b' and $-b''$ are mutually disjoint and disjoint from β, γ and δ. Also, we may assume that $-b'$, as a basing of the sublink $-L'$ of the split union, is disjoint from all other basings and bands. Invoke Lemma 5.5 to choose rationally slice links L_0, L_0' and L_0'' with basings b_0, b_0', b_0'' such that the connected sums $L \#_{(b,b_0)} L_0$, $L' \#_{(b',b_0')} L_0'$ and $L'' \#_{(b'',b_0'')} L_0''$ are in $\mathbb{W}_n$. Let

$$J' := \Bigg(\Big(\big((J \underset{(b,b_0)}{\#} L_0) \underset{(-b',-b_0')}{\#} -L_0'\big) \underset{(b',b_0')}{\#} L_0'\Big) \underset{(-b'',-b_0'')}{\#} -L_0''\Bigg).$$

Since L_0, L_0' and L_0'' are rationally slice, J' is rationally concordant to J. Define $L_1 := L \#_{(b,b_0)} L_0$, $L_1' := L' \#_{(b',b_0')} L_0'$ and $L_1'' := L'' \#_{(b'',b_0'')} L_0''$. Since our bands and basings are mutually disjoint, all the band sum and connect sum operations are associative. In particular, we have

$$J' = L_1 \underset{\beta}{\#} -L_1' \underset{\delta}{\#} L_1' \underset{\gamma}{\#} -L_1''.$$

Choose a basing c for L_1' to define a connected sum $-L_1' \#_{(-c,c)} L_1'$. Recall that α is the bands on $L \sqcup -L''$ given above. We may assume that both b for L and $-b''$ for $-L''$ have been chosen to be disjoint from α. Then, using α as bands on $L_1 \sqcup -L_1''$, a band sum $L_1 \#_\alpha -L_1''$ is defined. Choose a collection of bands ϵ to define

$$J'' := (-L_1' \underset{(-c,c)}{\#} L_1') \underset{\epsilon}{\#} (L_1 \underset{\alpha}{\#} -L_1'').$$

Since L_1, L_1', $-L_1'$ and L_1'' are in $\mathbb{W}_n$, a band sum of them is well-defined in $\mathsf{W}_n = \mathbb{W}_n/\mathbb{W}_{n+1}$, independent of the choice of the bands. Therefore, J' and J'' are order $n+1$

framed Whitney tower concordant in $S^3 \times I$. Since the connected sum $-L_1' \#_{(-c,c)} L_1'$ is slice in D^4, J'' is concordant to $L_1 \#_\alpha -L_1''$. Since

$$L_1 \underset{\alpha}{\#} -L_1'' = L_0 \underset{(b_0,b)}{\#} (L \underset{\alpha}{\#} -L'') \underset{(-b'',-b_0'')}{\#} -L_0'',$$

and since L_0 and L_0'' are rationally slice, $L_1 \#_\alpha -L_1''$ is rationally concordant to $L \#_\alpha -L''$. This completes the proof of the claim that $L \#_\alpha -L''$ is order $n+1$ framed Whitney tower concordant to J. $\square$

Let $\overline{\mathsf{W}}_n$ be the set of equivalence classes of links in $\overline{\mathbb{W}}_n$ under $\approx$. Denote by $[L] \in \overline{\mathsf{W}}_n$ the equivalence class of a link $L \in \overline{\mathbb{W}}_n$.

Theorem 5.9. *The band sum operation $[L] + [L'] := [L \#_\beta L']$ is well-defined on $\overline{\mathsf{W}}_n$, and $\overline{\mathsf{W}}_n$ is an abelian group under the band sum operation.*

Proof. Once we show that the band sum operation is well-defined, it follows immediately that $\overline{\mathsf{W}}_n$ is an abelian group; the identity is the class of a trivial link, and the inverse of $[L]$ is $[-L]$, the class of the mirror image of L with reversed orientation, since $L \#_{(b,-b)} -L$ is slice.

In what follows we will prove the well-definedness. Suppose $P \approx Q$ and $P' \approx Q'$ in $\overline{\mathbb{W}}_n$, that is, $P \#_\alpha -Q$ and $P' \#_{\alpha'} -Q'$ are in $\overline{\mathbb{W}}_{n+1}$ for some α and α'. We need to show that $P \#_\beta P' \approx Q \#_\gamma Q'$ for any given β and γ. We will proceed using essentially the same technique as that of the proof of Lemma 5.8.

Regard P, $-Q$, P' and $-Q'$ as sublinks of $P \sqcup -Q \sqcup P' \sqcup -Q'$, and choose δ disjoint from α and α' to define $P \#_\delta P'$. Then

$$J := (P \underset{\alpha}{\#} -Q) \underset{\delta}{\#} (P' \underset{\alpha'}{\#} -Q')$$

lies in $\overline{\mathbb{W}}_{n+1}$ since both $P \#_\alpha -Q$ and $P' \#_{\alpha'} -Q'$ are in $\overline{\mathbb{W}}_{n+1}$.

Choose basings b, $-c$, b' and $-c'$ of P, $-Q$, P' and $-Q'$ respectively, in such a way that they are mutually disjoint and are disjoint from α, α', β, γ and δ. Appealing to Theorem 4.5, choose rationally slice links P_0, Q_0, P_0' and Q_0' with basings b_0, c_0, b_0' and c_0' such that $P_1 := P \#_{(b,b_0)} P_0$, $Q_1 := Q \#_{(c,c_0)} Q_0$, $P_1' := P' \#_{(b',b_0')} P_0'$ and $Q_1' := Q' \#_{(c',c_0')} Q_0'$ are in $\mathbb{W}_n$. Then

$$J' := (P_1 \underset{\alpha}{\#} -Q_1) \underset{\delta}{\#} (P_1' \underset{\alpha'}{\#} -Q_1')$$

is rationally concordant to J.

Choose ϵ disjoint from b, b', $-c$, $-c'$, β and γ to define $P \#_\epsilon -Q$. Then

$$J'' := (P_1 \underset{\beta}{\#} P_1') \underset{\epsilon}{\#} -(Q_1 \underset{\gamma}{\#} Q_1')$$

is defined. Furthermore, since band sum is well-defined on $\mathsf{W}_n = \mathbb{W}_n/\mathbb{W}_{n+1}$ independent of the choice of bands and since P_1, Q_1, P_1', $Q_1' \in \mathbb{W}_n$, J'' is order $n+1$ framed Whitney tower concordant, in $S^3 \times I$, to J'. Since P_0, P_0', Q_0, Q_0' are rationally slice,

$$J''' := (P \underset{\beta}{\#} P') \underset{\epsilon}{\#} -(Q \underset{\gamma}{\#} Q')$$

is rationally concordant to J''. Combining the above, it follows that $J''' \in \overline{\mathbb{W}}_{n+1}$, that is, $P \#_\beta P' \approx Q \#_\gamma Q'$. $\square$

Now we compute the structure of $\overline{\mathsf{W}}_n$. Recall that $\mathcal{M}(m,n)$ is the number of linearly independent Milnor invariants of order n (see Remark 3.2).

Proof of Theorem 5.1. Since $\mu_n(L)$ vanishes for links L in $\overline{\mathbb{W}}_{n+1}$ by Theorem 5.6 and since μ_n is additive under band sum, μ_n induces a group homomorphism $\overline{\mathsf{W}}_n \to \mathsf{D}_n$. Recall that any class $[L] \in \overline{\mathsf{W}}_n$ is represented by a link $L \in \mathbb{W}_n$ by Theorem 5.6. It follows that the natural map $\mathbb{W}_n \to \overline{\mathbb{W}}_n$ and the induced homomorphism $\mathsf{W}_n \to \overline{\mathsf{W}}_n$ are surjective. Since

$$\begin{tikzcd} \mathsf{W}_n \arrow[rr, two heads] \arrow[dr, "\mu_n"'] & & \overline{\mathsf{W}}_n \arrow[dl, "\mu_n"] \\ & \mathsf{D}_n & \end{tikzcd}$$

is commutative, the image $\mu_n(\overline{\mathsf{W}}_n) \subset \mathsf{D}_n$ is equal to $\mu_n(\mathsf{W}_n)$. It is known that $\mu_n(\mathsf{W}_n)$ has the same rank as D_n, namely has rank $\mathcal{M}(m,n)$; for, since the realization $\widetilde{R}_n \colon \widetilde{\mathcal{T}}_n \to \mathsf{W}_n$ is surjective and $\mu_n(L) = \eta_n(t_n^{\infty}(T))$ for a bounding order n framed Whitney tower $T \subset D^4$ by [CST14, Theorem 6] or Theorem 3.1, $\mu_n(\mathsf{W}_n)$ is equal to the image of $\widetilde{\mathcal{T}}_n \to \mathcal{T}_n^{\infty} \to \mathsf{D}_n$, which has rank $\mathcal{M}(m,n)$ as stated in Section 5.1. This shows Theorem 5.1 (1).

For $n = 2\ell$, $[L] = 0$ in $\overline{\mathsf{W}}_n$ if and only if $\mu_n(L) = 0$, by Theorem 5.6. From this Theorem 5.1 (2) follows.

For $n = 2\ell - 1$, $\mathrm{SL}_{2\ell-1} = \mathrm{sl}_{2\ell} \circ \mu_{2\ell}$ on $\mathrm{Ker}\{\mu_{2\ell-1}\}$ is additive under band sum since so is $\mu_{2\ell}$. Therefore there is an induced homomorphism $\mathrm{SL}_{2\ell-1} \colon \mathrm{Ker}\{\mu_{2\ell-1}\} \to \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}$. This is an epimorphism. For, $\mu_{2\ell} \colon \overline{\mathsf{W}}^{\infty}_{2\ell} \to \mathsf{D}_{2\ell}$ is an isomorphism by Theorem 4.6 (2), and consequently the composition $\mathrm{sl}_{2\ell} \circ \mu_{2\ell} \colon \overline{\mathsf{W}}^{\infty}_{2\ell} \to \mathsf{D}_{2\ell} \to \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}$ with the quotient homomorphism $\mathrm{sl}_{2\ell}$ is surjective. Since every $L \in \overline{\mathbb{W}}^{\infty}_{2\ell}$ represents an element $[L] \in \mathrm{Ker}\{\mu_{2\ell-1}\} \subset \overline{\mathsf{W}}_{2\ell-1}$ and $\mathrm{SL}_{2\ell-1}(L) = \mathrm{sl}_{2\ell}(\mu_{2\ell}(L))$ by the definition of $\mathrm{SL}_{2\ell-1}$, it follows that $\mathrm{SL}_{2\ell-1} \colon \mathrm{Ker}\{\mu_{2\ell-1}\} \to \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}$ is surjective.

If $\mu_{2\ell-1}(L) = 0$, then $[L] = 0$ in $\overline{\mathsf{W}}_n$ if and only if $\mathrm{SL}_{2\ell-1}(L) = 0$ by Theorem 5.6. It follows that $\mathrm{SL}_{2\ell-1} \colon \mathrm{Ker}\{\mu_{2\ell-1}\} \to \mathbb{Z}_2 \otimes \mathsf{L}_{\ell+1}$ is injective. This completes the proof of Theorem 5.1 (3). □

References

[Cha07] Jae Choon Cha, *The structure of the rational concordance group of knots*, Mem. Amer. Math. Soc. **189** (2007), no. 885, x+95.

[Coc90] Tim D. Cochran, *Derivatives of links: Milnor's concordance invariants and Massey's products*, Mem. Amer. Math. Soc. **84** (1990), no. 427, x+73.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.

[CST07] James Conant, Rob Schneiderman, and Peter Teichner, *Jacobi identities in low-dimensional topology*, Compos. Math. **143** (2007), no. 3, 780–810.

[CST11] Jim Conant, Rob Schneiderman, and Peter Teichner, *Higher-order intersections in low-dimensional topology*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8131–8138.

[CST12a] James Conant, Rob Schneiderman, and Peter Teichner, *Tree homology and a conjecture of Levine*, Geom. Topol. **16** (2012), no. 1, 555–600.

[CST12b] ———, *Universal quadratic forms and Whitney tower intersection invariants*, Proceedings of the Freedman Fest, Geom. Topol. Monogr., vol. 18, Geom. Topol. Publ., Coventry, 2012, pp. 35–60.

[CST12c] ———, *Whitney tower concordance of classical links*, Geom. Topol. **16** (2012), no. 3, 1419–1479.

[CST14] J. Conant, R. Schneiderman, and P. Teichner, *Milnor invariants and twisted Whitney towers*, J. Topol. **7** (2014), no. 1, 187–224.

[CT04a] James Conant and Peter Teichner, *Grope cobordism and Feynman diagrams*, Math. Ann. **328** (2004), no. 1-2, 135–171.

[CT04b] ———, *Grope cobordism of classical knots*, Topology **43** (2004), no. 1, 119–156.

[Dwy75] William G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure Appl. Algebra **6** (1975), no. 2, 177–190.

[FQ90] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.

[FT95] Michael H. Freedman and Peter Teichner, 4*-manifold topology. II. Dwyer's filtration and surgery kernels*, Invent. Math. **122** (1995), no. 3, 531–557.

[IO01] Kiyoshi Igusa and Kent E. Orr, *Links, pictures and the homology of nilpotent groups*, Topology **40** (2001), no. 6, 1125–1166.

[Kru98] Vyacheslav S. Krushkal, *Additivity properties of Milnor's $\overline{\mu}$-invariants*, J. Knot Theory Ramifications **7** (1998), no. 5, 625–637.

[KT97] Vyacheslav S. Krushkal and Peter Teichner, *Alexander duality, gropes and link homotopy*, Geom. Topol. **1** (1997), 51–69 (electronic).

[Lev01] Jerome P. Levine, *Homology cylinders: an enlargement of the mapping class group*, Algebr. Geom. Topol. **1** (2001), 243–270 (electronic).

[Lev02] ———, *Addendum and correction to: "Homology cylinders: an enlargement of the mapping class group" [Algebr. Geom. Topol.* **1** *(2001), 243–270; MR1823501 (2002m:57020)]*, Algebr. Geom. Topol. **2** (2002), 1197–1204 (electronic).

[Mil57] John W. Milnor, *Isotopy of links. Algebraic geometry and topology*, A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, N. J., 1957, pp. 280–306.

[MKS66] Wilhelm Magnus, Abraham Karrass, and Donald Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966.

[Orr89] Kent E. Orr, *Homotopy invariants of links*, Invent. Math. **95** (1989), no. 2, 379–394.

[Sch06] Rob Schneiderman, *Whitney towers and gropes in 4-manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 10, 4251–4278 (electronic).

[ST04] Rob Schneiderman and Peter Teichner, *Whitney towers and the Kontsevich integral*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 101–134 (electronic).

[Sta65] John Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.

Department of Mathematics, POSTECH, Pohang Gyeongbuk 37673, Republic of Korea
School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
E-mail address: jccha@postech.ac.kr